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An Appraisal of the Efficiency of Alternative Deterministic Equivalents to the Stochastic Programming Model.

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ABSTRACT

The linear programming model with stochastic elements in the vector of cost coefficients or the vector of resource requirements has been approached in many ways. The foremost attempts at a solution involve the transformation of the model to a deterministic equivalent. There are a number of deterministic equivalents which have been developed for this purpose.

The objective of this study is to develop an experimental model which can be used to evaluate proposed deterministic equivalents to the stochastic programming model. This experimental model has been designed to determine the responses of a deterministic equivalent to induced changes in the properties and the positions of the stochastic parameters which appear in the linear programming model.

Three different linear deterministic equivalents were evaluated in this study. These were the one-stage expected value approach, the two-stage slack approach to programming under uncertainty, and the active approach to linear programming under risk.

The experimental model was used to evaluate, in turn, two different variations of an empirical stochastic

linear programming problem in terms of each deterministic equivalent. Two variations of the empirical problem were analyzed so that conclusions could be stated for either a tightly constrained or a slightly constrained problem. A Monte Carlo simulation of each of these empirical problems was also performed. The results of these simulations were used as standards with which to evaluate the results of each deterministic equivalent.

The experimental procedure was divided into three phases. In the first phase the stochastic parameters were limited to the vector of resource requirements, in the second phase the stochastic parameters appeared only in the vector of cost coefficients, while in the third phase the stochastic parameters appeared in both vectors simultaneously. In all cases the stochastic parameters were assumed to be normally and independently distributed with known means and variances, while the non stochastic parameters in the problem were assumed to be constant and equal to their expected values.

In all three phases of the experiment the deterministic equivalents were analyzed for each experimental problem as the positions of the stochastic parameters changed and as the variances of the stochastic parameters increased. For all initial conditions and for each of the deterministic equivalents, the null hypothesis of no difference between the results of the simulation approach and the results of the deterministic equivalent was tested at the levels of significance of $\alpha = .01$ and $\alpha = .05$.

In the first phase of the experiment an analysis of the results indicated that the two-stage slack approach yielded better results than either of the other deterministic equivalents evaluated. The results of the two-stage slack approach were generally feasible on the average, were not significantly different from the results of the simulation approach at either level of significance, and were consistent with respect to the two experimental problems considered. The expected value approach was found to yield the best results in phase two of the experiment. This approach yielded results which were generally feasible on the average, were not affected by the increases in the variances of the stochastic parameters, and were very reliable regardless of the type of problem analyzed. In the third phase the two-stage approach again yielded the best results. The results were generally feasible on the average and statistically the same as the results of the simulation approach at both levels of significance.

CHAPTER I

INTRODUCTION

Objective of the Study

The linear programming model with stochastic elements in the vector of cost coefficients or the vector of resource requirements has been approached in many ways. The foremost attempts at a solution involve the transformation of the model to a deterministic equivalent. There are a number of deterministic equivalents which have been developed for this purpose.

When the linear programming model containing stochastic parameters is transformed to a deterministic equivalent, then it is desirable to question the efficiency of the transformation used. In particular, three deterministic equivalents are evaluated. These are the one-stage expected value approach, the active approach to programming under risk, and the two-stage slack approach.

In order to determine the effectiveness of each deterministic equivalent, a Monte Carlo simulation of the stochastic model is used in this study as a standard for comparison. For each set of specifications of the model, the expected value of the optimal solutions derived from each deterministic equivalent is contrasted with the expected value of the optimal solutions derived from a simulation of the model. These expected values should not be considered optimal solutions in terms of the variables of the model. They are each estimates of the respective expected values of the optimal objective function values which are determined by randomly selecting from the distributions of the stochastic parameters specific values, which are then used to solve for a conditional optimal solution of each of the respective deterministic equivalents and of the simulation model.

The objective of this study is to determine the response of each of the types of deterministic equivalents indicated above to induced changes in the properties and the positions of the stochastic parameters which appear in the linear programming model. Each deterministic equivalent is to be evaluated under these changing conditions by utilizing the simulation solution as a standard. Specifically this study is concerned with the determination of the specific conditions under which any particular deterministic equivalent performs better than the others, and how much efficiency is lost in the application of each of these deterministic equivalents.

Although the primary objective of this study is the evaluation of those deterministic equivalents to the stochastic programming model which were mentioned above, it should also be pointed out that the development of an experimental model which can be used for this purpose is also an important result of this study. The reader should realize that the experimental model which was developed for this study is flexible in that it can be applied to the analysis of any proposed deterministic equivalent to the stochastic programming model.

Justification of the Study

The parameters of the linear programming model must be constant in order to use the simplex algorithm to correctly solve the model. The investigator would rarely meet a real world situation fulfilling this requirement for fixed parameters. Due to this fact the use of the simplex algorithm to solve most real world models is not theoretically justified. To overcome the problem created by the presence of the stochastic parameters, a deterministic equivalent to the model can be formulated and then solved.[1]

It should be understood that the use of a deterministic equivalent to solve a stochastic programming model is analogous to the use of the expected value of a

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[1]In this study only linear deterministic equivalents are considered. The reader should realize, however, that deterministic equivalents to the stochastic linear programming model can be non-linear. Some of these non-linear equivalents are also referred to in the next chapter.

variable to test an hypothesis or make a decision involving that variable. Use of the expected value of a variable in no way implies that the expected value completely describes the properties of the variable or, for that matter, the variable itself. The expected value of a variable is merely an efficient means of taking into account the influence of that variable in a deterministic decision-making procedure.

Similarly a deterministic equivalent can never be exactly the same as the stochastic model that it replaces. The deterministic equivalent represents an attempt to include in the deterministic solution procedure of a stochastic programming model the effects resulting from the presence of the variable parameters in that model.

The optimal solution of the model derived from a deterministic equivalent can only be considered to be an approximation to the true optimal solution of the stochastic model. The closeness of this approximation depends upon the deterministic equivalent which is used, the properties of the stochastic parameters, and the positions of these stochastic parameters in the model. The utilization of any particular deterministic equivalent should be investigated under changing conditions with respect to the properties and the positions of the stochastic parameters in the programming model.

If a relationship can be found between the properties and the positions of the stochastic parameters on the one hand and the closeness of the approximate solution to the true optimal solution on the other, then the investigator can use a particular deterministic equivalent with increased confidence. In effect, this relationship can be used to select the particular deterministic equivalent which minimizes the error incurred in approximating the true optimal solution to a stochastic programming model under a given set of initial conditions.

Scope and Limitations of the Study

The particular linear programming model utilized in this study to achieve the stated objective is an agricultural production model.[2] The stochastic parameters, when they appear in the model, are assumed to be independently distributed with normal distributions with known means and variances. The location of the stochastic parameters are restricted to the vector of resource requirements and the vector of profitability coefficients.

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[2]This model was formulated from data determined from an empirical study presented in M. M. Babbar, "Distributions of Solutions of a Set of Linear Equations (With an Application to Linear Programming)," Journal of the American Statistical Association, L (September, 1955), 854-869. This same problem was used to generate results for a study of linear programming under risk found in J. K. Sengupta and J. H. Portillo-Campbell, "A Fractile Approach to Linear Programming Under Risk," Management Science, XVI (January, 1970), 298-308.

It should be recognized that there are numerous ways in which the stochastic parameters can appear in the two vectors mentioned above. Specifically there are three general cases that can be identified. Stochastic parameters may appear in only the vector of resource requirements, only the vector of profitability coefficients, or in both vectors.

There are numerous ways in which the stochastic elements can appear in either of the vectors. For example, in the first general case all the parameters in the vector of resource requirements may be stochastic, or only some defined subset of these parameters may be stochastic. The same can be said for the vector of profitability coefficients. In the third general case, the matter is only compounded since various combinations of stochastic elements in both vectors must be considered.

In the case where there are n variables and m constraints in a model, then there are n elements in the vector of profitability coefficients and m elements in the vector of resource requirements. If only the vector of resource requirements is assumed to contain stochastic elements, then there are ${}_mC_i$ combinations in which i elements may be stochastic. The total number of ways in which stochastic elements may be combined in the vector of resource requirements is then $\sum_{i=1}^{m} {}_mC_i$.

Similarly there are $\sum_{i=1}^{n} {}_nC_i$ total ways in which stochastic elements can be combined in the vector of profitability coefficients and $\left(\sum_{i=1}^{m} {}_mC_i\right) \left(\sum_{i=1}^{n} {}_nC_i\right)$ total ways in which stochastic elements can be combined in both vectors simultaneously.

If the problem under investigation contains a large number of variables, a large number of constraints, or a large number of both variables and constraints; then an investigation of these three general cases would be quite lengthy. Because of this, the agricultural production model utilized in this study is restricted to a small number of both variables and constraints.

The justification for this restriction is reinforced by the fact that each initial formulation of the model can also be expanded. For example, once it has been determined which elements of the model are stochastic elements; then the properties of these stochastic elements can be changed. In this study only the effects induced by a change in the variances of the stochastic parameters are to be evaluated.

Outline of the Study

The formal presentation of the study is divided into five parts. Chapter one presents the objective, the justification, and the scope and limitation of the study. The theories of stochastic linear programming models are presented in chapter two. Special emphasis is placed upon the development of deterministic equivalents to these models. Chapter three discusses the requirements of simulation studies, with special emphasis given to the generation and testing of pseudorandom numbers. The particular deterministic equivalents which are tested in this study are highlighted in chapter four. This chapter includes a detailed statement of the experimental procedure used to achieve the stated objective. Chapter five is a summary of the results obtained from the simulation experiments and a statement of the conclusions drawn from this study.

CHAPTER II

STOCHASTIC LINEAR PROGRAMMING MODELS

Introduction

The initial development of the stochastic linear programming model is attributed to George Dantzig.[1] Since the appearance of this initial formulation, there have been many contributions made to the development of a theory of stochastic linear programming. It is necessary to assimulate this existing knowledge into a suitable format which can serve as a means of relating the results of the present study to the existing reservoir of understanding.

The objective of this chapter is to present the general theories of stochastic linear programming models. This objective can be accomplished through a dual classification system. The various stochastic linear programming models which have appeared in the literature and the various solution techniques which have been developed can both be classified. In attempting to classify the solution techniques special emphasis will be placed

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[1]George B. Dantzig, "Linear Programming Under Uncertainty," Management Science, I (April-July, 1955), 197-206.

upon the development of the different deterministic equivalents to the various stochastic linear programming models.

<u>General form of the stochastic linear programming model</u>

The generalized primal linear programming model (LP) can be formulated as follows:

$$\begin{aligned} &\text{Maximize:} && Z = C'X \quad , \\ &\text{subject to:} && AX \leq B \quad , \text{ and} \\ & && X \geq 0 \quad . \end{aligned} \qquad [1]$$

In this model C is a (n x 1) vector of profitability coefficients, A is a (m x n) matrix of technological coefficients, and B is a (m x 1) vector of resource restrictions. The dual associated with this model can be stated as:

$$\begin{aligned} &\text{Minimize:} && Z = B'W \quad , \\ &\text{subject to:} && A'W \geq C \quad , \text{ and} \\ & && W \geq 0 \quad . \end{aligned} \qquad [2]$$

The parameters A, B, and C in the models stated above are deterministic. In the primal model the set of inequalities form a convex polyhedral set over which the objective function is to be maximized. There are many solution procedures which can be used to find the maximum value of the objective function of the model. The simplex algorithm is one such procedure which is designed to move from one basic feasiable solution to the next while simultaneously increasing the value of the objective function along its searching path. Once the basic feasible solution, which maximizes the value of the objective function is found, the algorithm indicates that this maximum has been found. If the basis which maximizes the objective function contains m non-zero elements in the solution vector X then this is a nondegenerate basic feasible solution. If there are less than m non-zero elements in X, then the solution is degenerate. A similar presentation can be made for the dual problem.[2]

If some of the parameters of the LP model are considered to be stochastic, then the model fits the general description of a stochastic linear programming model (SLP). The SLP model takes account of the fact that there is a probability associated with each specific set (A, B, C) which forms the structure of the model. For example

$$P[A, B, C] = (A, B, C)_k = p_k \qquad [3]$$

where P stands for probability. It can be seen that:

$$p_k = [P(A = A_h) \cap P(B = B_i) \cap P(C = C_j)] \quad \text{, and}$$

$$\sum_k p_k = 1 \quad \text{, for all possible } k, \qquad [4]$$

where A_h, B_i, and C_j indicate specific values that these parameters can take on.

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[2]George Hadley, Linear Programming (Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1963), pp. 221-272.

The general formulation of the SLP model provides an appropriate starting point from which the different specific models can be deduced. These specific models can be classified under three broad headings:[3] chance-constrained programming, two-stage programming under uncertainty, and linear programming under risk.

The chance-constrained programming model[4] replaces the set of constraints of the LP model with a new set of conditions which can be stated as

$$P[AX \leqq B] \geqq \alpha \quad , \text{ and}$$
$$X \geqq 0 \qquad [5]$$

where P stands for probability and α is a (m x 1) column vector such that any particular α_i satisfies the condition $0 \leqq \alpha_i \leqq 1$. This vector contains a prescribed set of constants that are probability measures of the extent to which constraint violations are allowed.

The two-stage programming under uncertainty model[5] can be briefly stated as

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[3]J. K. Sengupta, G. Tintner, and C. Millham, "On Some Theorems of Stochastic Linear Programming with Applications," Management Science, X (October, 1963), 144-145.

[4]A. Charnes and W. W. Cooper, "Chance-Constrained Programming," Management Science, VI (October, 1959), 73-79.

[5]G. B. Dantzig and A. Madansky, "On the Solution of Two-Stage Linear Programs Under Uncertainty," in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, ed. by Jerzy Neyman (Berkeley: University of California Press, 1961), I, 165-176; and A. Madansky, "Dual Variables in Two-Stage Linear Programming Under Uncertainty," Journal of Mathematical Analysis and Applications, VI (February, 1963), 98-108.

$$\text{minimize:} \quad Z = C'X + E_{\min_y}[F'Y]$$

under the conditions

$$AX + DY \geq B \quad ,$$

$$X \geq 0, \; Y \geq 0 \quad . \qquad [6]$$

In this formulation the vector B contains random elements, E is the expected value operator, F and Y are (r x 1) column vectors, and D is a (m x r) matrix. This formulation introduces an additional variable Y into the model. The model in [6] must be minimized with respect to both X and Y.

The linear programming under risk classification encompasses all the approaches which are concerned with the statistical distribution of the objective function.[6] Models of this type consider the parameters of the LP model to be random variables with known probability distributions. Given this premise these models attempt either the optimization of the expected value of the objective function or the derivation of the statistical distribution of the objective function values.

Chance-Constrained Programming Models

Definition

In this section the chance-constrained interpretation

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[6] G. Tintner, "Stochastic Linear Programming with Applications to Agricultural Economics," in Proceedings of the Second Symposium on Linear Programming, ed. by H. A. Antosiewicz (Washington: National Bureau of Standards, 1955), I; and J. K. Sengupta, G. Tintner, and B. Morrison,

and its corresponding deterministic equivalents are defined. The different deterministic equivalents, related to chance-constrained programming, result from the different objectives which can be first formalized and then optimized by using these models.

A. Charnes and W. W. Cooper initially interpreted the SLP model as a chance-constrained model. They define the general class of SLP models in this way:

> The problem of stochastic (or better, chance-constrained) programming is here defined as follows: Select certain random variables as function of random variables with known distributions in such a manner as (a) to maximize a functional of both classes of random variables subject to (b) constraints on these variables which must be maintained at prescribed levels of probability. More loosely, the problem is to determine optimal stochastic decision rules under these circumstances.[7]

This definition equates all SLP models to chance-constrained programming models. The "optimal stochastic decision rule" mentioned in the definition refers to the transformation of the model to a specific deterministic equivalent which depends upon the form of the decision rule employed.

Charnes and Cooper are not alone in their formulation of deterministic equivalents to SLP models based upon a chance-constrained interpretation of the model.[8]

"Stochastic Linear Programming with Applications to Economic Models," Economica, XXX, No. 119 (1963), 262-276.

[7]Charnes and Cooper, "Chance-Constrained Programming," 73.

[8]Shinji Kataoka, "A Stochastic Programming Model," Econometrica, XXXI (January-April, 1963), 181-196; B. L. Millar; and H. M. Wagner, "Chance-Constrained Programming

The justification for this interpretation is that whenever stochastic parameters appear in a constraint of a programming model, then the question of that constraint being satisfied, once the optimal solution is found, can only be stated in probabilistic terms. H. Theil, for example, reports that "It is hardly reasonable to require that such an inequality [constraint] holds with certainty; indeed, it is much more reasonable to require that it holds with a sufficiently large probability."[9]

Charnes and Cooper emphasize that "...optimization under risk immediately raises very important questions concerning a choice of rational objectives."[10] In accordance with this feeling these authors examined three different types of objectives. These objectives include (1) an expected value optimization, (2) a minimum variance objective, and (3) a maximum probability model.

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with Joint Constraints," Operations Research, XXIII (November-December, 1965), 930-945; J. K. Sengupta, "Safety First Rules Under Chance-Constrained Linear Programming," Operations Research, XVII, (January-February, 1969), 112-132; Gifford H. Symonds, "Chance-Constrained Equivalents of Some Stochastic Programming Problems," Operations Research, XVI (November-December, 1968), 1152-1159; H. Theil, "Some Reflections on Static Programming Under Uncertainty," Weltwirtschaftliches Archiv, LXXXVII No. 1 (1961), 124-138; C. Van de Panne and W. Popp, "Minimum-Cost Cattle Feed Under Probabilistic Protein Constraints," Management Science, IX (April, 1963), 405-430.

[9]Theil, "Some Reflections on Static Programming," 124-125.

[10]A. Charnes and W. W. Cooper, "Deterministic Equivalents for Optimizing and Satisficing Under Chance Constraints," Operations Research, XI (January-February, 1963), 22.

These objectives are referred to respectively as (1) the E-model, (2) the V-model, and (3) the P-model.[11]

Regardless of the objective sought and the model which results from the formalization of that objective, all chance-constrained models can be transformed to some type of deterministic equivalent. The specific structure of this deterministic equivalent depends upon the choice of an objective and a suitable transformation.

The transformation is referred to as a decision rule. The decision rule is stated in terms of the parameters of the model, that is, the decision rule is some function of A, B, and C. The unknowns of the chance-constrained model are transformed according to this decision rule which can be stated generally as

$$X = \phi(A, B, C) \quad . \qquad [7]$$

The function, ϕ, should be chosen and applied in a manner that "...guarantees that the X values, as generated, will satisfy the chance constraints and optimize [the objective function]...."[12]

E-model[13]

The expected value model can be stated as

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[11] Ibid., 23.

[12] Ibid., 19-20. For a discussion of different classes of the decision rule see this source and also A. Charnes and M. J. L. Kirby, "Some Special P-Models on Chance-Constrained Programming," Management Science, XIV (November, 1967), 183-195.

[13] Charnes and Cooper, "Deterministic Equivalents," 25-30.

$$\text{Maximize:} \quad E[C'X]$$

$$\text{subject to:} \quad P[AX \leq B] \geq \alpha \quad ,$$

$$X = DB \qquad [8]$$

where the symbols have the same meanings as before. In the model E is the expected value operator and P is the symbol for probability. Charnes and Cooper consider A to be a matrix of known constants and B and C to be uncorrelated vectors each containing at least one random element.[14]

The last expression in the model in [8] indicates that a linear transformation of the variables in X is to be performed before a solution to the model is attempted. This transformation serves the purpose of converting a chance-constrained model, formulated in terms of the variables in X, into a deterministic equivalent in terms of the variables in D. The matrix D contains (n x m) unknowns which, when determined, are then used to specify the value of the variables in X.

Substituting the decision rule into the objective function of the model yields

$$E[C'X] = E[C'DB] = E[C']DE[B] \qquad [9]$$

If the E[C'] and the E[B] are defined respectively as μ_C' and μ_B, then the objective function can be written as

$$\text{Maximize:} \quad \mu_C' D\mu_B \quad . \qquad [10]$$

A consideration of the function in [10] reveals that this

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[14]*Ibid.*, 26.

function contains deterministic parameters with the variables being contained in the matrix D.

The conversion of the probabilistic constraints to corresponding deterministic constraints begins with substituting the function X = DB into the set of constraints. Making this substitution the constraint set can be written

$$P[ADB \leqq B] \geqq \alpha \quad . \qquad [11]$$

This set of constraints contains stochastic elements in the B vector. If a_i and b_i are the ith row of A and B respectively, then the ith constraint can be written

$$P[a_i DB \leqq b_i] \geqq \alpha_i \quad . \qquad [12]$$

If μ_B is defined as the mean vector of the parameters B, then the mean of the ith element, b_i, can be written μ_{b_i}. In addition $\hat{B}$ and $\hat{b}_i$ can be defined respectively as

$$\hat{B} = B - \mu_B$$

and $$\hat{b}_i = b_i - \mu_{b_i} \quad . \qquad [13]$$

The variate $(a_i DB - b_i)$, which is obtained from the left-hand side of the expression in [12], is assumed to be symmetrical and to be completely specified by its first two moments. Specifically this deviation is assumed to be normally distributed.[15] Given the assumption of symmetry the left-hand side of [12] can be written as

$$P[a_i DB - b_i \leqq 0] = P[b_i - a_i DB \geqq 0]. \qquad [14]$$

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[15]Ibid., 26-27.

Solving the expressions in [13] for B and b_i and substituting these results into [14] yields the following results

$$P[b_i - a_i DB \geqq 0] = P[\hat{b}_i + \mu_{b_i} - a_i D(\hat{B} + \mu_B) \geqq 0] \quad [15]$$

$$= P[\hat{b}_i - a_i D\hat{B} \geqq -\mu_{b_i} + a_i D\mu_B] \quad . \quad [16]$$

If $E[b_i - a_i DB]^2$ is assumed to be greater than zero,[16] then this expression can be divided into the two terms of the expression [16] to yield

$$P\left(\frac{\hat{b}_i - a_i D\hat{B}}{\sqrt{E[\hat{b}_i - a_i D\hat{B}]^2}} \geqq \frac{-\mu_{b_i} + a_i D\mu_B}{\sqrt{E[\hat{b}_i - a_i D\hat{B}]^2}} \right) \quad . \quad [17]$$

Upon inspection it can be seen that the left side of this expression is a standard deviate. By replacing the left side of the above expression with Z_i and then substituting the whole expression into [12] yields

$$P\left(Z_i \geqq \frac{-\mu_{b_i} + a_i D\mu_B}{\sqrt{E[\hat{b}_i - a_i D\hat{B}]^2}} \right) \geqq \alpha_i \quad . \quad [18]$$

This last expression can also be presented as

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[16] Ibid., 27. This assumption is made by the authors to simplify the derivation of the model.

$$F_i\left(\frac{-\mu_{b_i} + a_i D\mu_B}{\sqrt{E[\hat{b}_i - a_i D\hat{B}]^2}}\right) \geq \alpha_i \qquad [19]$$

where F_i is the cumulative distribution function of Z_i.

If α_i is assumed to be greater than 0.5,[17] then the expression on the right within the parentheses in [18] must be negative due to the properties of the standard normal deviate. This can be expressed as

$$\frac{-\mu_{b_i} + a_i D\mu_B}{\sqrt{E[\hat{b}_i - a_i D\hat{B}]^2}} \leq F_i^{-1}\,[\alpha_i] = -K_{\alpha_i} \qquad [20]$$

where F_i^{-1} is the inverse distribution function of the standard normal deviate for the ith constraint. In [20] K_{α_i} is a positive constant whose value can be determined given the probability level assigned to the ith constraint being satisfied.

In order to develop a deterministic equivalent which is a convex programming problem, each constraint in the constraint set [20] is first rewritten and then separated into an equivalent pair. Rewriting [20] yields

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[17] Ibid. This assumption is a realistic one when one considers practical applications relative to managerial policy problems. See for example: A. Charnes and W. W. Cooper, "Chance Constraints and Normal Deviates," Journal of the American Statistical Association, LVII (March, 1962), 134-148; and A. Charnes, W. W. Cooper, and G. H. Symonds, "Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil," Management Science, IV (April, 1958), 235-263.

$$-\mu_{b_i} + a_i D\mu_B \leqq -K_{\alpha_i} \sqrt{E[\hat{b}_i - a_i D\hat{B}]^2}$$

or

$$\mu_{b_i} - a_i D\mu_B \geqq K_{\alpha_i} \sqrt{E[\hat{b}_i - a_i D\hat{B}]^2} \quad . \qquad [21]$$

Since each term on the right hand side of this inequality is positive and their product is positive, then the expression on the left hand side must also be positive. Separating the set of constraints yields

$$\mu_{b_i} - a_i D\mu_B \geqq v_i$$

$$v_i \geqq K_{\alpha_i} \sqrt{E[\hat{b}_i - a_i D\hat{B}]^2} \quad . \qquad [22]$$

Upon separating the constraints a new set of variables is introduced into the model. The variable, v_i, serves the role of a slack variable for the ith constraint.[18] These slack variables are used to coordinate the two constraint sets in [22]. The first set of constraints in [22] is composed of the fixed parameters μ_{b_i}, a_i, and μ_B as well as the variables D and v_i. The second set of constraints contains the stochastic parameters $\hat{b}_i$ and $\hat{B}$ and the fixed element K_{α_i} in addition to the fixed parameter mentioned above. This second group of constraints incorporates the risk-taking elements of the original chance-constrained model into this determination equivalent; while the first group of constraints

[18] *Ibid.*, 29.

incorporates the original structural parameters into this deterministic equivalent.[19]

Since each term in the second expression in [22] is positive, the sense of the inequality is unaltered if each term in the inequality is squared. The resulting pair of constraint sets equivalent to the constraint sets is [21] is

$$\mu_{b_i} - a_i D\mu_B \geqq v_i$$
$$v_i^{\,2} \geqq K^2_{\alpha_i} \; E[\hat{b}_i - a_i D\hat{B}]^2 \quad . \qquad [23]$$

Rearranging the constraints in [23] yields

$$\mu_{b_i} - a_i D\mu_B - v_i \geqq 0$$
$$-K^2_{\alpha_i} \; E[\hat{b}_i - a_i D\hat{B}]^2 + \; v_i^2 \geqq 0 \quad . \qquad [24]$$

These constraints can be simplified by use of the following definitions

$$\mu_i[D] = (\mu_{b_i} - a_i D\mu_B) \quad , \text{ and}$$
$$\hat{\sigma}_i^2[D] = E[\hat{b}_i - a_i D\hat{B}]^2 \quad . \qquad [25]$$

The constraints of the deterministic equivalents can then be written[20]

$$\mu_i[D] - v_i \geqq 0$$
$$-K^2_{\alpha_i} \hat{\sigma}_i^2[D] + v_i^2 \geqq 0 \quad . \qquad [26]$$

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[19] Ibid.

[20] The presentation of the constraints in this form differs from the presentation of Charnes and Cooper. The

Each set of constraints above corresponds to a convex set; so that their intersection is also convex.[21]

The implication is that the deterministic equivalent to the E-model of a chance-constrained programming model is a convex programming problem. This deterministic equivalent can be written if the objective function [10] is combined with the set of constraints above. This yields

$$\begin{aligned} &\text{Maximize:} \quad \mu_C' D \mu_B \\ &\text{subject to:} \quad \mu_i[D] - v_i \geq 0 \quad , \\ &\qquad -K_{\alpha_i}^2 \hat{\sigma}_i^2[D] + v_i^2 \geq 0 \quad , \\ &\qquad v_i \geq 0 \quad . \end{aligned} \qquad [27]$$

This deterministic equivalent is stated in terms of the variables D and v_i where the d_{ij}'s are the structural variables and the v_i's are slack variables. The value of the slack variables can be increased so that the inequalities in [27] become equalities. In the special case when the elements of the B-vector are perfectly correlated, then the model above contains only linear constraints.[22] This would allow the use of the simplex algorithm in solving this model since the objective function of the model is also linear.

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constraints in this form can be applied directly to the solution of specific models. In their appendix the authors utilize the constraints in the form in which they are presented in this text.

[21]<u>Ibid.</u>, 28-29.

[22]<u>Ibid.</u>, 35-38.

V-model.[23]

The minimum variance model can be stated in the following way

$$\begin{aligned} \text{Minimize:} \quad & E[C'X - C^{o\prime}X^{o}]^2 \\ \text{subject to:} \quad & P[AX \leq B] \geq \alpha \quad , \\ & X = DB \quad . \end{aligned} \qquad [28]$$

As in the E-model, A is a matrix of known values and B and C are vectors which contain the stochastic elements. The effects, which result from a change in the objective of optimization, are incorporated only into the objective function of the model in [28]. This objective function states that the model seeks to minimize the squared deviation between the value $C'X$ and some desired value $C^{o\prime}X^{o}$, which is predetermined by the decision-maker.[24]

The derivation of the deterministic equivalent to this model is similar to the derivation of the deterministic equivalent for the expected value model, in view of the fact that the constraints of this model are the same as the constraints in the expected value model. By applying the decision rule $X = DB$ to the constraints in [28]; it can be seen that this operation should yield the same results that were determined in the expected

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[23] Ibid., 30.

[24] Ibid. See the footnote on this page for the interpretation to apply when μ_c' values are used in place of C'.

value model. The application of the decision rule in the objective function of [28] yields

$$\text{Minimize } E[C'DB - C^{o\prime}X^{o}]^2 \qquad [29]$$

which is the objective function of the deterministic equivalent to the V-model. If the expression in [29] is defined as V[D], then the deterministic equivalent to the V-model can be written

$$\begin{aligned} \text{Minimize:} \quad & V[D] \\ \text{subject to:} \quad & \mu_i[D] - v_i \geqq 0 \quad , \\ & -K_{\alpha_i}\hat{\sigma}_i^2[D] + v_i^2 \geqq 0 \quad , \\ & v_i \geqq 0 \quad . \end{aligned} \qquad [30]$$

This deterministic equivalent is a convex programming problem since these constraints are the same as those in [27]. In addition any differences in the d_{ij}'s and v_i's which result from the solution of the deterministic equivalents of the expected value and the minimum variance models are entirely due to difference in the functional forms of the objective function of the two models.

<u>P-model.</u>[25]

In this model the objective is to maximize the probability of achieving some specified $C^{o\prime}X^{o}$, which is determined from the aspiration level of the decision-maker. This model can be formalized as

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[25] *Ibid.*, 30-33. See also Charnes and Kirby, "Some Special P-Models," 183-195.

$$\begin{aligned} \text{Maximize:} \quad & P[C'X \geq C^{o\prime}X^o] \\ \text{subject to:} \quad & P[AX \leq B] \geq \alpha \quad , \\ & X = DB \quad . \end{aligned} \qquad [31]$$

By utilizing the decision rule X = DB and transforming this model in the same manner as the E and V-models were transformed, the deterministic equivalent for the P-model can be written

$$\begin{aligned} \text{Maximize:} \quad & v_o/w_o \\ \text{subject to:} \quad & \mu_C' D \mu_B - v_o \geq \mu_{C^o} \quad , \\ & -V[D] + w_o^2 \geq 0 \quad , \\ & \mu_i[D] - v_i \geq 0 \quad , \\ -K_{\alpha_i} & \hat{\sigma}_i^2[D] + v_i^2 \geq 0 \quad , \\ & v_i \geq 0 \quad . \end{aligned} \qquad [32]$$

The last three constraints in this deterministic equivalent are the same constraints that appear in the expected value and the minimum variance models. In conjunction with these, two additional constraints result from the derivation. These additional constraints constitute the objective functions of both of the previous deterministic equivalents. The v's and w's which appear in both the objective function and the constraints are slack variables.

The objective function of this model is stated as a fractional and assumes "...a minimax-like character in the sense that maximization of v_o/w_o represents a striving toward cooperatively maximizing v_o while minimizing w_o."[26] Since the constraints still form a convex set and since the objective function is a fractional, the following transformation can be performed to replace the fractional programming problem with a simple convex programming problem.[27] Assuming $w_o > 0$, define a variable t, which is used to transform the variables of the fractional programming problem, as follows

$$\bar{w}_o = tw_o = 1 \quad . \qquad [33]$$

The remaining variables D and v are transformed by the following relationships

$$\bar{D} = tD \quad , \text{ and}$$
$$\bar{v} = tv \quad . \qquad [34]$$

Substituting in [32] for w_o, D, and v the above transformation yields the following convex programming problem

$$\begin{aligned} \text{Maximize:} \quad & \bar{v}_o \\ \text{subject to:} \quad & \mu_C'\bar{D}\mu_B - \bar{v}_o \geqq t\mu_C^o \quad , \\ & -\bar{V}[\bar{D}] + \bar{w}_o^2 \geqq 0 \quad , \\ & \bar{\mu}_i[\bar{D}] + \bar{v}_i \geqq 0 \quad , \\ & -K_{\alpha_i}\hat{\sigma}_i^2[\bar{D}] + \bar{v}_i^2 \geqq 0 \quad , \\ & \bar{w}_o = 1 \quad , \\ & t, \bar{v}_i \geqq 0 \quad , \qquad [35] \end{aligned}$$

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[26] Charnes and Cooper, "Deterministic Equivalent," 32.

[27] A. Charnes and W. W. Cooper, "Programming with Linear Fractional Functionals," <u>Naval Research Logistics</u>

where

$$\overline{V}[\overline{D}] = E[C'\overline{D}B - tC^{o\prime}X^{o}]^2 \quad ,$$

$$\hat{\sigma}_i^2[\overline{D}] = E[t\hat{b}_i - a_i\overline{D}\hat{B}]^2$$

$$\overline{\mu}_i[\overline{D}] = (t\mu_{b_i} - a_i\overline{D}\mu_B) \quad . \qquad [36]$$

This last model in [35] is the convex programming problem which is the deterministic equivalent to the P-model of a chance-constrained programming problem.

The foregoing should not be interpreted as a complete presentation of deterministic equivalents to chance-constrained models. The reader should consider the fact that the deterministic equivalents which result depend upon the type of decision rule which is used to transform the original model. Only a linear decision rule was considered in this discussion. Various transformations that can be used to derive deterministic equivalents to chance-constrained models have appeared in the literature.[28] These other transformations are not considered

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<u>Quarterly</u>, IX (September-December, 1962), 181-186. In this source theorems can be found which establish the criteria for converting a fractional functional programming problem to a convex programming problem.

[28] In addition to references made in other parts of this chapter, the reader can also consider: A. Charnes, W. W. Cooper, and G. L. Thompson, "Constrained Generalized Medians and Hypermedians As Deterministic Equivalents for Two-Stage Linear Programs Under Uncertainty," <u>Management Science</u>, XXII (September, 1965), 83-112; A. Charnes, M. J. L. Kirby, and W. M. Raike, "Solution Theorems In Probabilistic Programming: A Linear Programming Approach," <u>Journal of Mathematical Analysis and Applications</u>, XX

here since the purpose of this discussion has been primarily to classify chance-constrained models according to the objective functions used in the model.

Two-Stage Programming Under Uncertainty

The two-stage programming model refers to all stochastic programming models which allow adjustments to be made once the stochastic elements of the model have been observed to be equal to specific values. These adjustments are made by including in the stochastic programming model a new variable which attempts to compensate for infeasible solutions which result from the previous actions of the decision-maker. The solution of a stochastic programming model by utilizing a two-stage approach has been referred to as the slack solution.[29]

Slack solution.[30]

The slack solution can be explained if one considers the linear programming model

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(December, 1967), 565-582; Fredrik S. Hillier, "Chance-Constrained Programming With 0-1 or Bounded Continuous Decision Variables," Management Science, XIV (September, 1967), 34-57; and Gifford H. Symonds, "Deterministic Solutions For A Class of Chance Constrained Programming Problems," Operations Research, XV (May-June, 1967), 495-512.

[29] A. Madansky, "Methods of Solution of Linear Programs Under Uncertainty," Operations Research, X (July-August, 1962), 463-471.

[30] Ibid., 468-470.

$$\text{Minimize:} \quad C'X$$
$$\text{subject to:} \ AX \geq B \quad ,$$
$$X \geq 0 \qquad [37]$$

In this model, if the A matrix and the B vector were to contain stochastic elements, then the possibility would arise that an optimal solution to the model could violate some of the constraints of the model. This possibility is dependent upon the subsequent observations on the elements of the A matrix and the B vector.

Instead of minimizing the objective function C'X over the convex set defined by A and B, the two-stage solution procedure allows an adjustment to be made after calculating X and subsequently observing A and B. This adjustment for the possible infeasibility of a selected X is in the form of a new variable, Y, with a corresponding penalty cost given by F'Y, where F is a vector of penalty cost coefficients. Both F and Y are (m x 1) vectors corresponding to the dimensions of B. The choice of the vector Y depends not only on the original stochastic parameters A and B but also upon the initial solution vector X. In view of the inclusion of this new variable, Y, the objective function of the resultant model must also be adjusted to take into consideration both the cost, C'X, and the penalty cost, F'Y, which may be incurred.

This two-stage programming model is a special case[31] of a general class of programming models which can be stated

$$\begin{aligned} \text{Minimize}_X: \quad & E_{\min_Y}[C'X + F'Y] \\ \text{subject to:} \quad & AX + DY = B \quad , \\ & X \geqq 0 \quad , \\ & Y \geqq 0 \quad . \end{aligned} \qquad [38]$$

In this general model A and D are (m x n) matrices, C, X, F, and Y are (n x 1) vectors, and B is a (m x 1) vector composed of stochastic elements with known distributions. The objective function is composed of two types of cost, the cost associated with each element of X and the penalty cost associated with each element of Y.

The general model in [38] can be specialized to the two-stage programming model by considering DY to be equivalent to $(Y^+ - Y^-)$. The vector Y that yields the smallest penalty cost for each A, B, and X would then be composed of two parts.

$$\begin{aligned} \text{If} \quad & B \geqq AX, \text{ then} \\ & Y^+ = B - AX \text{ and} \\ & Y^- = 0 \quad . \end{aligned} \qquad [39]$$

[31] Ibid., 468. The reader can also consider the treatment presented in Dantzig and Madansky, "Solution of Two Stage Linear Programs," 165-166.

If $B < AX$, then

$$Y^{+} = 0 \quad \text{and}$$

$$Y^{-} = AX - B \quad . \qquad [40]$$

In some cases some of the rows of A and B may not be stochastic. If this is the case then the corresponding constraints contain no Y elements. These constraints are then called fixed constraints on X.[32]

The objective function of the general model in [38] can also be specialized to accommodate the two-stage programming model. Since the choice of Y depends upon both B and X, the objective function can be formulated to minimize C'X plus the expected smallest penalty cost.

The two-stage programming model can then be written:

$$\text{Minimize:} \quad C'X + E_{\min_Y}[F'Y]$$

$$\text{subject to:} \quad AX + (Y^{+} - Y^{-}) = B \quad ,$$

$$X \geqq 0, \quad Y \geqq 0 \qquad [41]$$

where Y^{+} and Y^{-} are defined in [39] and [40]. From the format it can be seen that the Y's act as slack variables either reducing the left side of the equation when infeasibility occurs or increasing the left side of the equation when the initial solution X does not utilize the total resources available in B. If the ith row of A and B contain deterministic elements, then the ith constraint is a fixed constraint and can therefore be written without the variable Y.

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[32] Madansky, "Methods of Solutions," 469.

The assumption implied in the model above is that for each $X \geq 0$, which satisfies all the constraints, and for each B there exists a Y such that (X, Y) is feasible. As an alternative to this assumption define "...K as the convex set of the X's such that each $X \epsilon K$ is nonnegative and has an associated Y for each A and B such that (X, Y) is feasible. The problem is, then, to find $X \epsilon K$ that minimizes $C'X + E_{\min_Y} [F'Y]$."[33]

Linear Programming Under Risk

Those models that are classified under this general heading can be grouped into two distinct classes. The first class contains one-stage linear programming models under uncertainty. There are two different solution procedures that can be used to solve these models. These procedures are called the expected value solution and the "fat" solution. The second class contains the models that are formulated to specify the statistical distribution of the objective function of a stochastic model. These models can be formulated in terms of either an active or a passive approach to the problem. Whether an active or passive approach is used to specify the statistical distribution of the objective function, these models assume that the distributions of the parameters of the stochastic model are known. In effect the models which

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[33] Ibid., The reader can also consult Dantzig and Madansky, "Solution of Two Stage Linear Programs," 166.

fall into this category are referred to as linear programming models under risk.

One-stage model[34]

The expected value solution

The implication of an expected value solution to a one-stage stochastic linear programming model can best be explained if first a deterministic linear programming model is stated in terms of a matrix game. Consider the deterministic model

$$\begin{aligned} \text{Minimize:} \quad & C'X \\ \text{subject to:} \quad & AX \geqq B \quad , \\ & X \geqq 0 \quad . \end{aligned} \qquad [42]$$

This model is feasible and finite "...if and only if the matrix game with payoff matrix

$$Q = \begin{vmatrix} 0 & A & -B \\ -A' & 0 & C \\ B' & -C' & 0 \end{vmatrix} \qquad [43]$$

has an optimal mixed strategy (X_o', Y_o', t) such that $t > 0$."[35] Under these conditions the solutions to the primal and the dual model are given by

$$\begin{aligned} X &= X_o/t \quad \text{and} \\ Y &= Y_o/t \quad . \end{aligned} \qquad [44]$$

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[34] Madansky, "Methods of Solutions," 464-468.

[35] Ibid., 464-465. The reader should also consider the reference given in the cited text.

When the parameters of the programming model are stochastic and an expected value solution to that model is attempted, then the corresponding matrix game has a payoff matrix E[Q] with an optimal strategy given by $(\overline{Y}', \overline{X}', \overline{t})$ where $\overline{t} > 0$ and E is the expected value operator. In this case the expected value solution of the model is $X^* = \overline{X}/\overline{t}$. This solution minimizes the model

$$\begin{aligned} \text{Minimize:} \quad & E[C']X \\ \text{subject to:} \quad & E[A]X \geqq E[B] \quad , \\ & X \geqq 0 \quad . \end{aligned} \qquad [45]$$

The solution vector X^* is nonnegative, but may not guarantee that the constraint set in [42] is satisfied. Let S be defined as the set of permanently feasible X's, that is

$$\{X \varepsilon S: \quad X \geqq 0, \Pr[AX \geqq B] = 1\} \quad . \qquad [46]$$

Now if (E[A], E[B]) is a member of the set of values (A, B) and if X^* is permanently feasible, then X^* is a solution to the stochastic programming model.

The necessary and sufficient conditions for the expected value solution X^* to be an optimal solution can be determined from the payoff matrix given in [43] if the following definitions are made:

$$\begin{aligned} \overline{Z}' &= (\overline{Y}', \overline{X}', \overline{t}) \quad \text{and} \\ M[Q] &= Q\overline{Z}' \quad . \end{aligned} \qquad [47]$$

The term $\overline{Z}'$ is defined as the optimal set of strategies for the matrix game E[Q], and the term M[Q] is the product of the original payoff matrix and the vector comprising the optimal strategies.

The first m-rows of the matrix M[Q] can be written $(A\overline{X} - B\overline{t})$. These rows correspond to the constraints of the primal model stated in [42]. Since $\overline{t} > 0$, then

$$(A\overline{X} - B\overline{t}) \geqq 0 \quad \text{if and only if}$$
$$A\overline{X}/\overline{t} = AX^* \geqq B \qquad [48]$$

for all A and B. The conclusion is that the expected value solution is optimal, if and only if, for all values of A and B, the first m-rows of M[Q] are nonnegative.

If the ith row is one of the first m-rows of M[Q], then this row can be written

$$M_i[Q] = a_{i1}\overline{X}_1 + a_{i2}\overline{X}_2 + \ldots + a_{in}\overline{X}_n - b_i\overline{t}. \qquad [49]$$

The necessary and sufficient conditions for the expected value solution to be optimal are satisfied if the minimum of $M_i[Q]$ with respect to the elements $(a_{i1}, \ldots, a_{in}, b_i)$ is greater than or equal to zero.[36]

The "fat" solution

The "fat" solution was initially proposed as a means of accounting for uncertainties which may develop in the long-run when a deterministic solution to a programming

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[36] Ibid., 466. Madansky assumes that the set of possible values $(a_{i1}, \ldots, a_{in}, b_i)$ form a bounded convex polyhedron and that the minimum of $M_i[Q]$ is taken subject to that condition.

model was the basis upon which decisions were made.[37] The procedure which would yield a "fat" solution involves ignoring the random variation and providing plenty of "fat" in the deterministic version of the model. Consider the model stated in [42] where now the parameters (A, B) are stochastic. Utilizing the "fat" solution one would postulate a pessimistic (A, B) and then solve the non-stochastic program. The choice of the appropriate pessimistic values of the stochastic elements should be such that the optimal solution to the program is from the set of permanently feasible X's.[38] In the situation where there are a finite number of possible (A, B)'s, the set of permanently feasible X's can be described as those X's that satisfy the mR constraints

$$A^{(r)}X \geqq B^{(r)} \quad \text{for } r = 1, \ldots, R, \text{ and}$$
$$X \geqq 0 \quad . \qquad [50]$$

The optimal solution of the stochastic program is the X, from the set of permanently feasible X's, which minimizes C'X subject to the constraints stated in [50] above. In the case where C contains stochastic elements, the function to be minimized can be stated as E[C']X.

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[37] George B. Dantzig, "Recent Advances in Linear Programming," Management Science, II (January, 1966), 131; Salah E. Elmaghraby, "An Approach to Linear Programming Under Uncertainty," Operations Research, VII (March-April, 1959), 208-209.

[38] Madansky, "Methods of Solution," 467.

Distribution models

The approach taken toward stochastic linear programming models which concerns itself with the specification of the distribution of the objective function was introduced by Gerhard Tintner. This approach is based upon the assumption that the parameters of a linear programming model are random variables with known probability distributions. These models have been described in the following way.

> If all the parameters in a linear programming problem are random variables, the problem becomes a stochastic programming problem. A passive solution exists if the activities are not chosen in advance. We have an active solution if the proportion of the resources to be devoted to various activities are chosen. By numerical methods we can determine approximations to the various distributions and choose the optimal one [solution] according to some criteria.[39]

It can be seen that the distribution model, specified by Tintner, is approached from two different points of view. It is the objective of this section to specify both the passive and the active approaches to the distribution model.[40]

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[39]G. Tintner, "A Note on Stochastic Linear Programming," Econometrica, XXVIII (April, 1960), 490.

[40]These two approaches are specified in Sengupta, Tintner, and Morrison, "Stochastic Linear Programming with Applications," 262-276; J. K. Sengupta, G. Tintner, and G. Millham, "On Some Theorems of Stochastic Linear Programming with Applications," Management Science, X (October, 1963), 143-159; and K. D. Cocks, "Discrete Stochastic Programming," Management Science, XV (September, 1968), 72-79.

The passive approach

In the passive approach the objective is to determine the expected value and variance of the distribution of the optimal objective function values. This can be formally stated as: Find the $E[Z(X)]$ and $V[Z(X)]$ in the model

$$\begin{aligned} \text{Maximize:} \quad & Z_k = C_k X_k \\ \text{subject to:} \quad & A_k X_k \leqq B_k \quad , \\ & X_k \geqq 0 \quad , \text{ for } k = 1, \ldots, N. \end{aligned} \qquad [51]$$

The parameters A, B, and C are randomly distributed with known distributions, such that A_k, B_k, and C_k are specific values that each respective parameter may take on. The specific value of each parameter is determined a priori from the characteristics of its distribution and substituted into the model in [51] to determine an optimal value of the objective function Z_k. If this process is repeated N times, with N optimal values of the objective function being determined, then the expected value and variance of the optimal objective function values can be defined as

$$\begin{aligned} E[Z(X)] &= \sum^{N} Z(X)_k P[Z(X)_k] \\ V[Z(X)] &= \sum^{N} \{Z(X)_k - E[Z(X)_k]\}^2 P[Z(X)_k] \quad , \end{aligned} \qquad [52]$$

where

$$P[Z(X)_k] = P[A = A_k \cap B = B_k \cap C = C_k] \quad . \qquad [53]$$

This appraoch assumes "...that in almost all possible situations, i.e., for almost all possible variations of the parameters, the conditions of the simple nonstochastic linear program are fulfilled and the maximum achieved."[41]

The active approach

The model[42] utilized in the active approach can be formalized as

$$\begin{aligned} \text{Maximize:} \quad & Z = C'X \\ \text{subject to:} \quad & A\hat{X} \leqq \hat{B}\,U \quad , \\ & X \geqq 0 \quad . \end{aligned} \qquad [54]$$

In this model U is a (m x n) matrix with the elements u_{ij} satisfying the conditions

$$u_{ij} \geqq 0 \quad , \text{ and}$$
$$\sum_{j=1}^{n} u_{ij} = 1 \quad . \qquad [55]$$

Further, X and B are square, diagonal matrices with the elements of the X and B vectors making up the respective diagonals of these matrices. The vector of cost coefficients is C.

The u_{ij}'s are allocation ratios such that u_{ij} indicates the proportion of the resource i which is used for activity j. These allocation ratios are exogenous variables and are determined by the policies of the decision-maker. The only conditions placed upon these ratios are those stated in [55] above which imply that all resources must be completely used up.

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[41] Sengupta, Tintner, and Millham," Theorems of Stochastic Linear Programming," 145.

[42] Ibid.

In the active approach the objective is also to determine the probability distribution of the optimal values of the objective function given the alternative values that can be assumed by the parameters A, B, and C. In this case, however, the distribution of the optimal values of the objective function is also dependent upon the allocation matrix U. The implication is that a set of distributions of optimal objective function values results from the set of allocation matrices that are employed. In effect the decision-maker can measure the effects of various policy decisions upon the distributions of the optimal objective function values by simply changing the matrix U.

A specific decision rule must be established in order to distinguish among the different distributions which can result due to the employment of alternative sets of allocation ratios. Three different decision rules or criteria of optimization under risk have been utilized to distinguish among the distributions explained above. These are the expected value criteria, the fractile criteria, and the portfolio or aspirations level criteria.[43]

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[43]These three criteria are specified for the case where the C vector is randomly distributed in the following: J. K. Sengupta and J. H. Portillo-Campbell, "A Fractile Approach to Linear Programming Under Risk," Management Science, XVI (January, 1970), 298-308; and A. M. Geoffrion, "Stochastic Programming with Aspiration or Fractile Criteria," Management Science, XIII (May, 1967), 672-679.

Expected value criteria.--Under the expected value criteria the choice among the distributions of the optimal objective function values is made by selecting the distribution with the maximum expected value. If K different allocation matrices are considered, then the maximum value in the set

$$\{E[Z(X)]_{U=U_1}, \quad E[Z(X)]_{U=U_2}, \quad \ldots, E[Z(X)]_{U=U_k}\} \qquad [56]$$

is selected. This selection indicates the set of allocation ratios which optimizes the stochastic model.

Fractile criteria.--The fractile criteria specifies that the α-fractile of the distribution of optimal objective function values is to be optimized. Selecting the maximum value from the set

$$\{F_\alpha[Z]_{U=U_1}, \quad F_\alpha[Z]_{U=U_2}, \quad \ldots, F_\alpha[Z]_{U=U_k}\} \qquad [57]$$

satisfies this decision rule. In this relation the α is a predetermined constant such that $F_\alpha[Z]$ is the α-fractile of the distribution of optimal objective function values.

Portfolio criteria.--The application of the portfolio criteria requires that the variance of the distribution of optimal objective function values is minimized under the additional constraint that the expected values of these same distributions are at least equal to some preassigned level. This criteria involves selecting the minimum from the set

$$\{V[Z(X)]_{U=U_1}, \quad V[Z(X)]_{U=U_2}, \quad \ldots, V[Z(X)]_{U=U_R}\} \qquad [58]$$

subject to the additional constraint that the

$$E[Z(X)]_{U=U_1}, \ldots, E[Z(X)]_{U=U_R} \geqq P_o$$

where

$$0 \leqq R \leqq K \quad \text{, and} \qquad [59]$$

P_o is a predetermined profit level.

The fractile criteria possesses an important characteristic.[44] Consider the case when $\alpha = 0.5$. In this situation the fractile criteria yields the same results as the expected value criteria. Similarly, an appropriate value of α can be determined such that the results from the application of the fractile criteria and the portfolio criteria are the same. In effect the expected value criteria and the portfolio criteria are special cases of the fractile criteria.

Computationally the fractile criteria is difficult to apply since for the most part the objective function is nonlinear. Consider the case where $\alpha > 0.5$ and only the C vector is stochastic, having a multivariate normal

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[44]Sengupta and Portillo-Campbell, "A Fractile Approach Under Risk," 299-300.

distribution with mean M and variance V, the model can then be written[45]

$$\text{Maximize:} \quad M'X - q(X'VX)^{1/2}$$

$$\text{subject to:} \quad AX \leq B \quad ,$$

$$X \geq 0 \quad , \qquad [60]$$

where q is a positive standard normal deviate and M'X and (X'VX) are the mean and variance of the objective function Z = C'X. Iterative algorithms have been developed to analytically solve a model of this type.[46]

Summary

The purpose of this chapter has been to classify into groups the stochastic linear programming models which have appeared in the literature. These groups consist of (1) the chance-constrained programming models, (2) the two-stage programming under uncertainty model, and (3) the linear programming under risk model. Within each classification various solution procedures or deterministic equivalents to these models were then indicated. The E-model, the V-model, and P-model were shown to be deterministic equivalents to the chance-constrained programming models. The two-stage programming under uncertainty model was approached through a slack solution.

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[45] Ibid., 299.

[46] Ibid., 300-301. See also: Geöffrion, "Stochastic Programming with Aspiration Criteria," 672-679: and Kataoka, "A Stochastic Programming Model," 181-196.

And the linear programming under risk models were classified as either one-stage models or distribution models.

The reader should realize that, even though the classification scheme above is all-inclusive, not all possible solution techniques or deterministic equivalents to these models have been presented. The emphasis in this chapter has been upon the classification of the models. The more important solution techniques have been included in order to enhance this objective.

CHAPTER III

REQUIREMENTS FOR SIMULATION STUDIES

Introduction

The experimental results of this study are generated by using a simulation procedure. A discussion of the Monte Carlo simulation technique is therefore a necessary prerequisite to the presentation of the experimental model used in this study. This chapter meets this objective by defining simulation and briefly presenting the properties and characteristics of Monte Carlo simulation models. Special emphasis is placed upon the development and the testing of the pseudorandom number generator which is used in this study.

Operations research models can utilize different problem-solving procedures. These are (1) the analytical, (2) the numerical, and (3) the simulation procedures.[1]

The analytical procedure is based upon mathematical deduction. The decision-maker works from a set of defined assumptions adhering to all the rules of mathematical logic until the solution is derived. The expression of the

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[1]W. W. Thompson, Operations Research Techniques (Columbus, Ohio: Charles E. Merrill Books, 1967), pp. 4-6.

model in a mathematically rigid way is the essential prerequisite for using an analytical procedure. In effect the relationships among the variables in the model must be both identified and rigidly defined.

The numerical procedure is one in which the assumptions and the relationships among the variables of the model are exactly defined, but the solution to the model is obtained through a less formal trial-and-error technique. The use of the numerical procedure is restricted to those models where either no analytical procedure is applicable or where the analytical procedure is too inconvenient to apply.

A simulation procedure is used when the model of the system is too complex for the effective use of either of the two other procedures. In the simulation procedure a set of synthetic variables, representing an analogous set of real world variables, is manipulated with the purpose of arriving at conclusions about the real world system being studied. The first step of this procedure is the representation of the real world situation in the form of an abstracted model. The model must then be manipulated in order to generate a set of synthetic outputs which are characteristic of the real world system.

Monte Carlo Simulation

Simulation has been described as an experimental technique involving logical and mathematical models of a real world system. The experimentation is performed under stochastic or dynamic conditions. In addition the experimental results from a simulation experiment which is run on a computer may not necessarily be determinable by analytical methods.[2]

The essence of the above statements is that simulation is a special kind of experimentation. Specifically it is mathematical experimentation. Simulation has been considered a form of mathematical experimentation by many authors. For example, J. M. Hammersly and D. C. Handscomb classify simulation as a tool of experimental or theoretical mathematics which relies upon the deductive process.[3] James R. Jackson concludes that

> Mathematical experimentation [simulation] may be an appropriate research technique when interesting problems appear too difficult for the effective application of the traditional deductive approach. From a mathematical point of view, the conclusions reached can rarely be thought of as more than conjectures; but it seems to me that when a high degree of confidence can be placed in experimentally reached conclusions, they are of virtually the same practical interest as would be proven theorems with the same content.... It is important to note that I am proposing simulation experimentation as a supplement to mathematical analysis, not as a substitute therefor.[4]

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[2]Thomas H. Naylor, et al., Computer Simulation Techniques (New York: John Wiley and Sons, 1966), pp. 2-3.

[3]J. M. Hammersly and D. C. Handscomb, Monte Carlo Methods (London: Methuen, 1964), p. 1.

[4]James R. Jackson, "Simulation as Experimental Mathematics," in Symposium on Simulation Models: Methodology and Applications to the Behavioral Sciences ed. by Austin C. Hoggatt and Frederick E. Balderston (Cincinnati, Ohio: South-Western Publishing Co., 1963), p. 246.

Monte Carlo methods comprise that segment of simulation techniques which is concerned with experiments having a stochastic or probabilistic structure. The stochastic components of the system are included in the model through the use of a random number generator. Monte Carlo simulation is an efficient means of analyzing and solving stochastic models when these models are considerably complex. In some cases a Monte Carlo simulation may be the only means available for analyzing and solving a stochastic model.

The justification for using a Monte Carlo simulation to derive the results of this study is based upon the size and the objective of the study. With regard to size, the reader should recall the numerous initial formulations of the stochastic linear programming model which can be established. Each of these initial formulations is then considered under varying assumptions concerning the stochastic elements and in terms of the different deterministic equivalents to the stochastic model. A Monte Carlo simulation can efficiently analyze such a large scale model.

The objective of this study is the evaluation of the performance of the different deterministic equivalents under varying conditions. Monte Carlo simulation is an efficient means of analyzing the internal interactions among the components of a model when the assumptions and, or, the parameters of the model are allowed to change. The most important feature of Monte Carlo simulation is the flexibility which it allows the researcher in manipulating the components of a model.

Random Number Generators[5]

A Monte Carlo simulation is as valid as the random number generator which is used to produce the values of the stochastic elements which are integral parts of the simulation. For each simulation it is important to select a random number generator which has been properly tested. The verification of the random number generator is an essential part of any simulation experiment.

This section contains a general discussion of random number generators with emphasis upon the congruential methods of generating random numbers. The specific random number generator used in the simulation experiment described in this work is explained at the end of this section. The verification of this generator is the topic of the next section of this chapter.

There are many alternative methods available for generating random numbers. This section is limited to a discussion of digital computer methods for their generation. Digital computer methods provide for the internal generation of a sequence of digits by a recurrence relationship. The immediate advantage of this procedure

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[5]Naylor, Computer Simulation Techniques, pp. 43-67.

is the small amount of computer memory required to perform the operation. An additional advantage is that the process is totally reproducible.

The use of a recurrence relationship may appear to be in conflict with the randomness required of the digits in the sequence. Since each digit in the sequence can be determined from the previous digit or from some set of previous digits through the recurrence relationship, then the process is technically not random. The process is defined as random if and only if the sequence meets certain statistical tests of randomness. If these statistical tests are met, the sequence is called a series of pseudorandom numbers.

The desirable properties of a sequence of pseudorandom numbers generated internally on a digital computer are that the numbers be (1) uniformly distributed, (2) statistically independent, (3) reproducible, and (4) nonrepeating for a sufficient length. This last property refers to the period of the sequence of numbers. In addition the generator should require a minimum amount of computer memory and should generate the pseudorandom numbers at a high rate of speed.[6]

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[6] _Ibid._, p. 46.

Congruential Methods for Generating Pseudorandom Numbers

Most computer codes for generating random numbers use some variation of the congruential method developed by D. H. Lehmer.[7] The congruential method is speedy, reproducible, and requires only a small amount of computer memory. The period of the sequence of random numbers depends upon the particular congruential method used. Whatever congruential method is used statistical tests should be performed on the sequence of numbers to determine whether they are uniformly distributed and statistically independent. Most congruential generators satisfy the statistical tests of uniformity and independence.

Fundamental congruential relationship

Congruential methods are based on a fundamental congruential relationship. This relationship can be expressed in the following way.

> Two integers a and b are congruent modulo m if their difference is an integral multiple of m. The congruence relation is expressed by the notation $a \equiv b \pmod{m}$ which reads "a is congruent to b modulo m."[8]

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[7] D. H. Lehmer, "Mathematical Methods in Large-Scale Computing Units," Annals Computer Laboratory Harvard University, XXVI (1951), 141-146.

[8] Naylor, Computer Simulation Techniques, p. 64.

The implications of this definition are (1) that $(a - b)$ is divisible by m and (2) that a and b leave identical remainders when divided by the absolute value of m. This relationship can be expressed as the following recursive formula

$$n_{i+1} \equiv an_i + c \qquad (\text{mod } m) \quad , \qquad [1]$$

where n_i, a, c, and m are all nonnegative integers.[9] Given a constant multiplier a and an additive constant c, this formula establishes the relationship between any number in a sequence and the previous number.

The period, h, of the above formula is the length of the sequence before a number repeats itself, that is, before $n_h = n_o$. Once a number in the sequence is repeated, then the series will duplicate itself; that is, $n_{h+1} = n_1$, $n_{h+2} = n_2$, etc. Theorems are available to show that the congruential methods have a finite period which depends upon the constants in the recursive formula in [1].[10]

Basic types of congruential generators

Three congruential methods have been developed for generating pseudorandom numbers. Each of these methods is based upon the relation in [1]. Each method

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[9] Ibid., p. 48.

[10] Ibid., pp. 65-66. The reader should also refer to M. D. MacLaren and G. Marsaglia, "Uniform Random Number Generators," *Journal of the Association for Computing Machinery*, XII, (January, 1965), 86-89.

is designed to generate a sequence of pseudorandom numbers with a maximum period in a minimum amount of time. These methods are (1) the additive congruential method, (2) the multiplicative congruential method, and (3) the mixed congruential method.

Additive congruential method[11]

The additive congruential method assumes that k random numbers are provided in computer memory. The sequence of pseudorandom numbers is computed by means of the congruence relationship

$$n_{i+1} \equiv n_i + n_{i-k} \pmod{m} \quad . \qquad [2]$$

The pseudorandom numbers generated in this way have a period which depends upon k and m. Statistical tests have indicated that k = 16 is the smallest value which will yield acceptable random numbers.[12]

Multiplicative congruential method[13]

The multiplicative congruential method is based upon the congruence relationship

$$n_{i+1} \equiv an_i \pmod{m} \quad , \qquad [3]$$

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[11] Naylor, Computer Simulation Techniques, p. 49, 56-57.

[12] B. F. Green, J. Smith, and L. Klem, "Empirical Tests of an Additive Random Number Generator," Journal of the Association for Computing Machinery, VI (October,1959), 537.

[13] Naylor, Computer Simulation Techniques, pp. 49, 51-54.

where a is a positive constant. This relationship yields a sequence of positive integers less than m. The period of the sequence depends upon both the constant multiplier a and the initial value in the series n_o. Conditions can be placed upon these constants to insure a maximum period. Statistical tests have been performed which indicate that the multiplicative congruential method generally yields a sequence of pseudorandom numbers which is uniformly distributed and statistically independent.[14]

Mixed congruential method[15]

The mixed congruential method is based upon the recursive relationship as it is expressed in [1] with both a and c not equal to zero. With this method the maximum period depends upon the constants a and c. Little or no effect upon the statistical properties of the sequence is attributed to the initial value n_o.[16]

Some investigators have expressed dissatisfaction with the congruential methods discussed above. The essence of this criticism is the failure of the congruential methods to always satisfy the requirement for serial

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[14]MacLaren, "Uniform Random Number Generators," 86-89; and T. E. Hull and A. R. Dobell, "Random Number Generators," SIAM Review, IV (July, 1962) 238-242.

[15]Naylor, Computer Simulation Techniques, pp. 49, 55-56.

[16]J. L. Allard, A. R. Dobell, and T. E. Hull, "Mixed Congruential Random Number Generators for Decimal Machines," Journal of the Association for Computing Machinery, X (April, 1963), 131-141.

independence.[17] Two remedies have been proposed to alleviate this problem. R. R. Coveyou and M. Greenberger have determined theoretical conditions on the values of a, c, and m in the fundamental congruential relationship in [1] which will guarantee a small serial correlation among the numbers in the sequence. These empirical studies specify that a value of $a = \sqrt{m}$ will yield the smallest value for the correlation coefficient regardless of the value of c.[18] The second remedy proposed by M. D. MacLaren and G. Marsaglia is called the combination method. With this procedure a mixed congruential generator is used to randomly determine an index which is used to select a random number from a set of stored random numbers. The stored random numbers are generated by the multiplicative congruential method such that the ith number is replaced by a new value when i is the index generated by the mixed congruential method.[19]

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[17] MacLaren, "Uniform Random Number Generators," pp. 86-89.

[18] R. R. Coveyou, "Serial Correlation in the Generation of Pseudo-Random Numbers," Journal of the Association for Computing Machinery, VII (January, 1960), 72-74; and M. Greenberger, "An a Priori Determination of Serial Correlation in Computer Generated Random Numbers," Mathematics of Computations, XV (October, 1961), 384-386.

[19] MacLaren, "Uniform Random Number Generators," pp. 83-86.

<u>The random number generator used in this study</u>

The random number generator used in the simulation experiment of this study is shown in the appendix to this chapter. The generator is a multiplicative congruential generator with multiple initial values. This generator is recommended for use on the IBM 360 computer.[20] Random real numbers between zero and one and random integers between zero and 2^{31} are computed with this generator. The random integer produced at any stage is used as the input value for determining the random number at the next stage. The period of this generator is $h = 2^{29}$. The multiple initial values have been added to the generator by this researcher in order to reduce the serial correlation among the numbers in the sequence.

Recalling the multiplicative congruential relationship

$$n_{i+1} \equiv an_i \qquad (\text{mod } m) \quad , \qquad [4]$$

it is necessary to specify the values of the constants in that relationship which guarantee a maximum period and a minimum value of the serial correlation coefficient for the sequence of numbers generated by the relationship. The condition on the multiplicative constant a is that

[20] This generator can be found in the <u>IBM Application Program, System 360 Scientific Subroutine Package</u>, (360A - (m - 03X) Version III, p. 77. The reader is also referred to <u>IBM Manual C20-8011</u> on random number generators.

it must be odd and relatively prime to m.[21] Since the IBM 360 computer is a binary computer $m = 2^b$ where b is the number of binary digits in a word. (The value of b is thirty-one for the IBM 360 computer.) The values of a which satisfy the condition above can be expressed as

$$a = 8t \pm 3 \quad , \tag{5}$$

where t is any positive integer.[22]

According to the conditions set down by Coveyou and Greenberger, values of a should be chosen to minimize the first order serial correlation of the numbers in the sequence. A value of a approximately equal to $\sqrt{m}$, which is equal to $2^{b/2}$ in this case, should be chosen. The value of a recommended with this generator is 65539. It was found that this value does not satisfy the condition for minimizing the serial correlation of the sequence. As an alternative to the above integer the value 46331 is used with the generator since this integer satisfies all the conditions stated above.

The condition on the initial value, n_o, specifies that it be any positive odd number. To meet the design of this particular random number generator eleven initial values of n_o are stored in computer memory. The first ten

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[21] Naylor, Computer Simulation Techniques, pp. 63-64. Two integers are relatively prime if the greatest common divisor of the two integers is one.

[22] Ibid., pp. 51-52.

of these values are the multiple starting values to be used to generate a pseudorandom number. The eleventh value is used to randomly determine an index which indicates which starting value is to be used to determine that pseudorandom number. Each time that a particular starting value is used it is updated, that is the ith pseudorandom number in integer form serves as the starting value for the generation of the next pseudorandom number. The eleven values of n_o used with this generator are shown in the appendix.

Verification of the Random Number Generator

Three types of statistical tests are used to verify the random number generator described at the end of the last section. These types of tests are (1) a uniform frequency test, (2) a serial correlation test, and (3) a test of runs.[23] First, second, and third order serial correlation tests are performed on the sequence of pseudorandom numbers generated. Two types of runs tests are performed. These involve runs above and below the mean and runs up and down. The FORTRAN program written to perform these tests and the results of the tests are presented in the appendix to this chapter.

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[23]Ibid., pp. 57-62. The tests described in this section are based upon the corresponding tests presented in this reference.

Each test is designed to be run on a predetermined number of groups with each group containing a predetermined number of pseudorandom numbers. The purpose for arranging and testing the random numbers in groups is that the consistency of the generator with respect to its meeting a particular test can be observed. In addition subsequent groups, which were not originally planned for, can be tested along with the initial groups at a later time without having to again generate the initial groups.

Uniform Frequency Test

The uniform frequency test is a chi-square test used to test whether the sequence of pseudorandom numbers is uniformly distributed. The test is performed on a sequence of AM consecutive sets of AN pseudorandom numbers.[24]

The generator produces real numbers between zero and one. This interval is divided into ten subintervals. The expected number of pseudorandom numbers in each group which falls into each subinterval is AN divided by ten. Let f_j denote the actual number of pseudorandom numbers in the subinterval

$$(j - 1)/10 \leq r < j/10, \text{ where } j = 1,2,\ldots,10 \qquad [6]$$

[24] In explaining the logic behind these tests the variable names used in the FORTRAN program are incorporated into the text whenever such an inclusion is beneficial to the reader.

The statistic

$$\chi_1^2 = \sum_{j=1}^{10} \frac{(f_j - \frac{AN}{10})^2}{\frac{AN}{10}} = \frac{10}{AN} \sum_{j=1}^{10} (f_j - \frac{AN}{10})^2 \qquad [7]$$

is then distributed according to a chi-square distribution with nine degrees of freedom.[25]

This chi-square statistics is computed for each of the AM groups of AN pseudorandom numbers. These AM values of χ_1^2 are then grouped into four intervals in the following way. Let F_i be the number of the resulting values of χ_1^2 which lie between the (i - 1)th and the ith quartile of a chi-square distribution with nine degrees of freedom. Since the expected number of χ_1^2 values which fall in each interval is AM divided by four, then the statistic

$$\chi_F^2 = \frac{4}{AM} \sum_{i=1}^{4} (F_i - \frac{AM}{4})^2 \qquad [8]$$

is chi-square distributed with three degrees of freedom. The hypothesis concerning the randomness of the sequence of pseudorandom numbers is rejected if the statistic χ_F^2 is greater than a critical value of the chi-square statistic with three degrees of freedom. The critical

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[25] In this statistic and in all the others to be explained the values of AM and AN must be chosen so as to guarantee that the expected number of elements falling into each subinterval is greater than five. This is based upon the assumptions under which a chi-square test is performed.

value is determined by assuming a given level of significance at which the hypothesis is tested.

Serial Correlation Test

Serial correlation tests are used to determine the independence of successive pseudorandom numbers in a sequence. First, second, and third order serial correlation tests are run. These tests respectively determine the independence between the ith and the $(i + h)$th pseudorandom number in the sequence. The value h equals one for the first order test, two for the second order test, and three for the third order test.

The serial correlation tests used in this study are also based upon a chi-square distribution. For each set of AN pseudorandom numbers let f_{jk} denote the number of pseudorandom numbers which fall in the intervals

$$(j - 1)/10 \leq r_i < j/10 \quad \text{and}$$

$$(k - 1)/10 \leq r_{i+h} < k/10 \qquad [9]$$

where the intervals are the same as those used in the uniform frequency test. In the expressions above j and k range from one to ten, i ranges from one to $(AN - h)$, and h equals one, two, or three respectively for the first, second, or third order serial correlation test.[26]

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[26] For the sake of clarity, the remainder of the discussion is in terms of only one serial correlation test. To perform the three tests the only change in procedure involves using a different value of h.

The statistic

$$\chi_2^2 = \frac{10^2}{AN - h} \sum_{j=1}^{10} \sum_{k=1}^{10} (f_{jk} - \frac{AN - h}{10^2})^2 \qquad [10]$$

(for h = 1, 2, 3)

is chi-square distributed with ninety-nine degrees of freedom since there are one hundred classes into which a pair of successive pseudorandom numbers can fall. This statistic is determined for each of the AM groups of pseudorandom numbers.

The statistic $(\chi_2^2 - \chi_1^2)$ is calculated for each of the AM groups of pseudorandom numbers. This statistic is distributed according to a chi-square distribution with ninety (100 - 10) degrees of freedom.[27] Let s_j denote the number of values of $(\chi_2^2 - \chi_1^2)$ which lie between the (j - 1) th and the jth quartile of a chi-square distribution with ninety degrees of freedom. Then the statistic

$$\chi_S^2 = \frac{4}{AM} \sum_{j=1}^{4} (s_j - \frac{AM}{4})^2 \qquad [11]$$

is chi-square distributed with three degrees of freedom. The serial independence of the sequence of pseudorandom numbers is established at a given level of significance if the values χ_F^2 and χ_S^2 are each less than the critical

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[27] Ibid., p. 59. The reader is also referred to I. J. Good, "On the Serial Test for Random Sequences," Annals of Mathematical Statistics, XXVIII (March, 1957), 262-264.

value of the chi-square statistic with three degrees of freedom.

Runs Tests

Runs tests are concerned with the particular arrangement of the pseudorandom numbers within the sequence. Since the pseudorandom numbers should be uniformly distributed, the mean of the numbers should equal the median of the numbers.

The test for runs above and below the mean is designed to result in a rejection of the hypothesis of randomness if the sequence of numbers is such that any number, which is greater than (or less than) the mean or median of the numbers in the sequence, is repeatedly followed by a number which is also greater than (or less than) the mean or median of the numbers in the sequence. If the generator is producing pseudorandom numbers then the conditional probability that some number r_{i+1}, is greater than the mean, given that r_i is greater than the mean, is equal to the probability that r_{i+1} is less than the mean given that r_i is greater than the mean. A similar equality holds for the conditional probabilities in the case when r_i is less than the mean.

A similar argument can be made concerning the test for runs up and down. In this case the hypothesis of randomness is rejected if the numbers in the sequence are repeatedly larger (or smaller) than the previous numbers.

Runs above and below the mean

Since the generator produces pseudorandom numbers over the interval from 0 to 1, then the mean of the pseudorandom numbers should be 0.5. Each of the AN pseudorandom numbers in each group can be classified as either greater than, less than, or equal to 0.5. For each sequence of AN pseudorandom numbers a corresponding sequence can be constructed. If $r_i < 0.5$, define $s_i = 0$; and if $r_i > 0.5$, define $s_i = 1$. Values of r_i exactly equal to 0.5 are improbable and are not counted as a run. The runs in s_i are accumulated by size and are then compared with the expected number of runs of each size. The expected number of total runs is $(AN + 1)/2$ and the expected number of runs of any length k is

$$\frac{(AN - k + 3)}{2^{k+1}} \quad . \qquad [12]$$

The chi-square statistic for this test can be written

$$\chi^2_{A/B} = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E} \qquad [13]$$

where O_i is the observed number of runs of a given size i and E_i is the expected number of runs of size i.

The larger the size of a run the smaller is the probability of a run of that size. Therefore the expected number of runs of fairly large size is small. Care must be taken in performing this chi-square test to include in the test only terms whose expected values exceed five. Whenever for a given run size the expected number of runs of that size is less than five, then this run size and all larger run sizes must be grouped into a single class in order for the chi-square test to be performed correctly. The last term in the test compares the observed and expected numbers of runs of some size j and all sizes larger than j. The largest run size which has an expected value greater than or equal to five depends upon AN, the number of pseudorandom numbers generated in a group.

Once the number of terms to be included in the chi-square test is known, then the degrees of freedom for the test is one less than the number of terms. All the statistical tests in this study were performed on 30 groups of 400 pseudorandom numbers each. That is AM = 30 and AN = 400. In the test of runs above and below the mean, run sizes as large as five had expected values greater than five. The last term of the chi-square test then compared the observed and expected number of runs of sizes six or more. The degrees of freedom then associated with this test is five.

The statistic $\chi^2_{A/B}$ must be determined for all AM groups of pseudorandom numbers. Let s_j denote the number of the resulting $\chi^2_{A/B}$ values that lie between the (j - 1)th and the jth quartile of a chi-square distribution with five degrees of freedom. The statistic

$$\chi^2 = \frac{4}{AM} \sum_{j=1}^{4} (s_j - \frac{AM}{4})^2 \qquad [14]$$

is then chi-square distributed with three degrees of freedom. The test is met at a given level of significance if this value is less than a critical value of chi-square with three degrees of freedom.

Runs up and down

The test for runs up and down is analogous to the test for runs above and below the mean. For each sequence of AN pseudorandom numbers again a corresponding sequence can be constructed. If a particular pseudorandom number r_i is less than the next number in the sequence r_{i+1} then $s_i = 0$. When r_i is greater than r_{i+1}, then $s_i = 1$. The runs in s_i are then accumulated by size. The expected number of total runs is (2AN - 1)/3; the expected number of runs of length k is

$$\frac{2[(k^2 + 3k + 1)AN - (k^3 + 3k^2 - k - 4)]}{(k + 3)!} \qquad [15]$$

for k less than (AN - 1); and the expected number of runs of length (AN - 1) is (2/AN!).

The statistic $\chi^2_{U/D}$ can be defined in a similar fashion as the statistic $\chi^2_{A/B}$ in [13]. Again care must be taken to include in this test only run sizes whose expected numbers are greater than five. With AM = 30 and AN = 400, run sizes up to a length of three have expected values greater than five. The last term in the test therefore compares the observed and expected numbers of run sizes of four or more. The degrees of freedom of the statistic $\chi^2_{U/D}$ is three.

The statistic $\chi^2_{U/D}$ is determined for each of the AM groups of pseudorandom numbers. Again the number of the values of $\chi^2_{U/D}$ which fall into the various quartiles of a chi-square distribution with three degrees of freedom are determined. A statistic analogous to that in [14] is then found and the test is completed in a similar fashion.

<u>The test results</u>

The sequence of pseudorandom numbers is tested by arranging the sequence into 30 (AM) groups with 400 (AN) pseudorandom numbers in each group. The sequence numbers of the pseudorandom numbers in the first set of the 30 groups run from 1 to 12,000. The test results shown in the appendix were obtained by running and testing 12 sets of groups. The total number of pseudorandom numbers tested is 144,000.

The hypothesis of randomness is tested at a level of significance of $\alpha = .05$ and $\alpha = .01$. The critical values of the chi-square statistic with three degrees of freedom at these levels of significance are respectively 7.81473 and 11.3449.

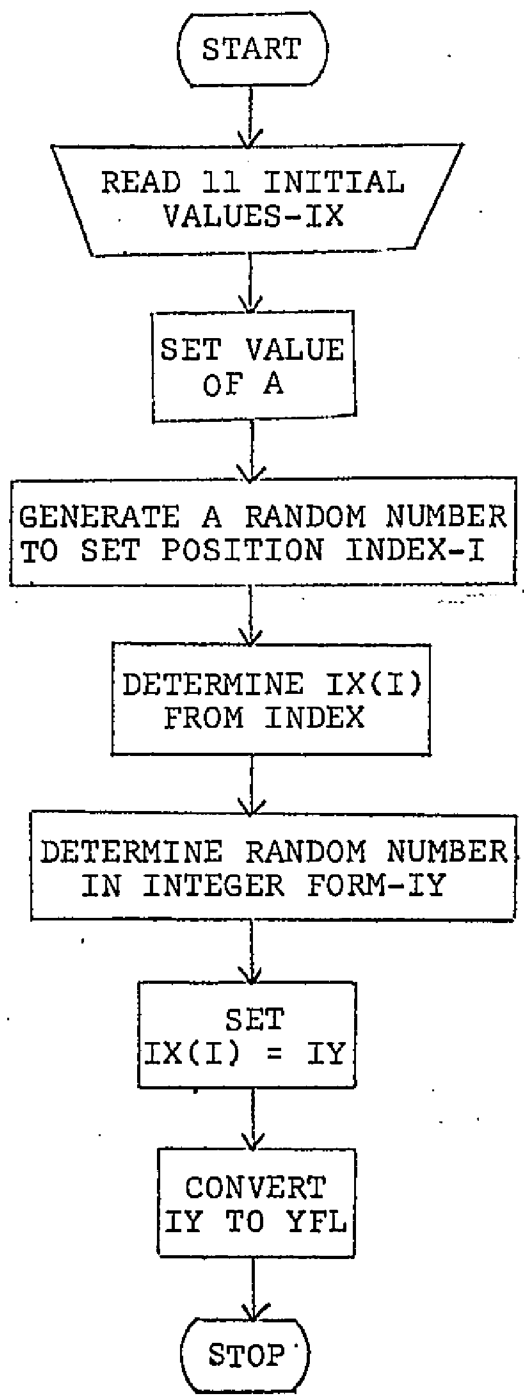
The results of these different tests are presented in Table 1 in the appendix. The generator passed all tests of randomness at a level of significance of $\alpha = .01$. At the significance level of $\alpha = .05$ the generator failed the various tests on three occasions. The first order serial correlation test led to a rejection of the hypothesis of independence for the set of pseudorandom numbers with sequence numbers from 120,001 to 132,000. Similarly the second order serial correlation test led to a rejection of the hypothesis for the numbers whose sequence numbers are from 108,001 to 120,000. On one occasion the test of runs above and below the mean led to a rejection of the hypothesis of randomness. This occurred for the pseudorandom numbers with sequence numbers from 132,001 to 144,000.

The failure to accept the hypothesis on these three occasions at the level of significance of $\alpha = .05$ does not lessen the confidence that this generator is producing a sequence of numbers which can be assumed to be random. These three failures represent a smaller proportion of failures than that considered acceptable at the 0.5 level of significance.

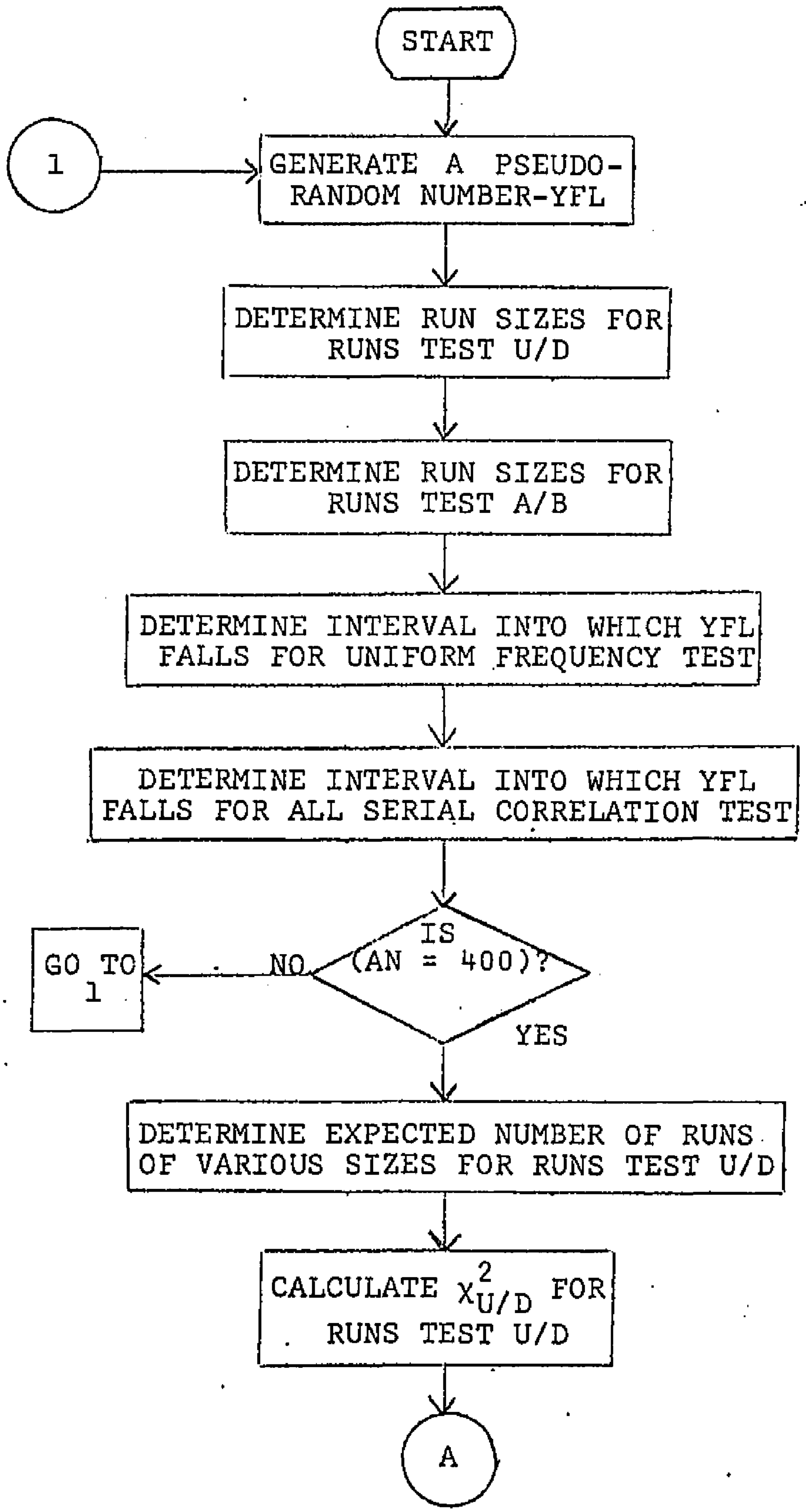
APPENDIX

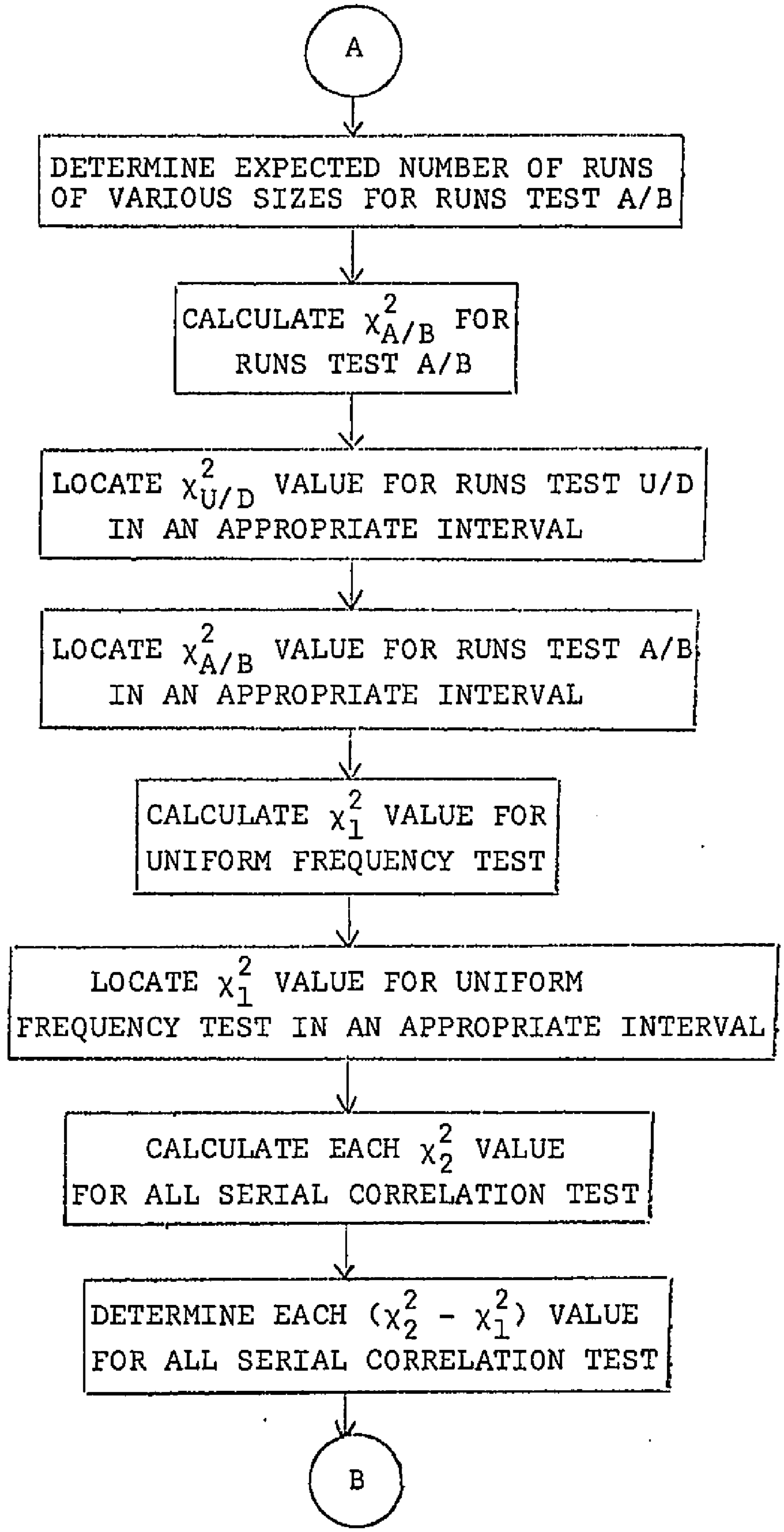
FLOW CHART OF THE

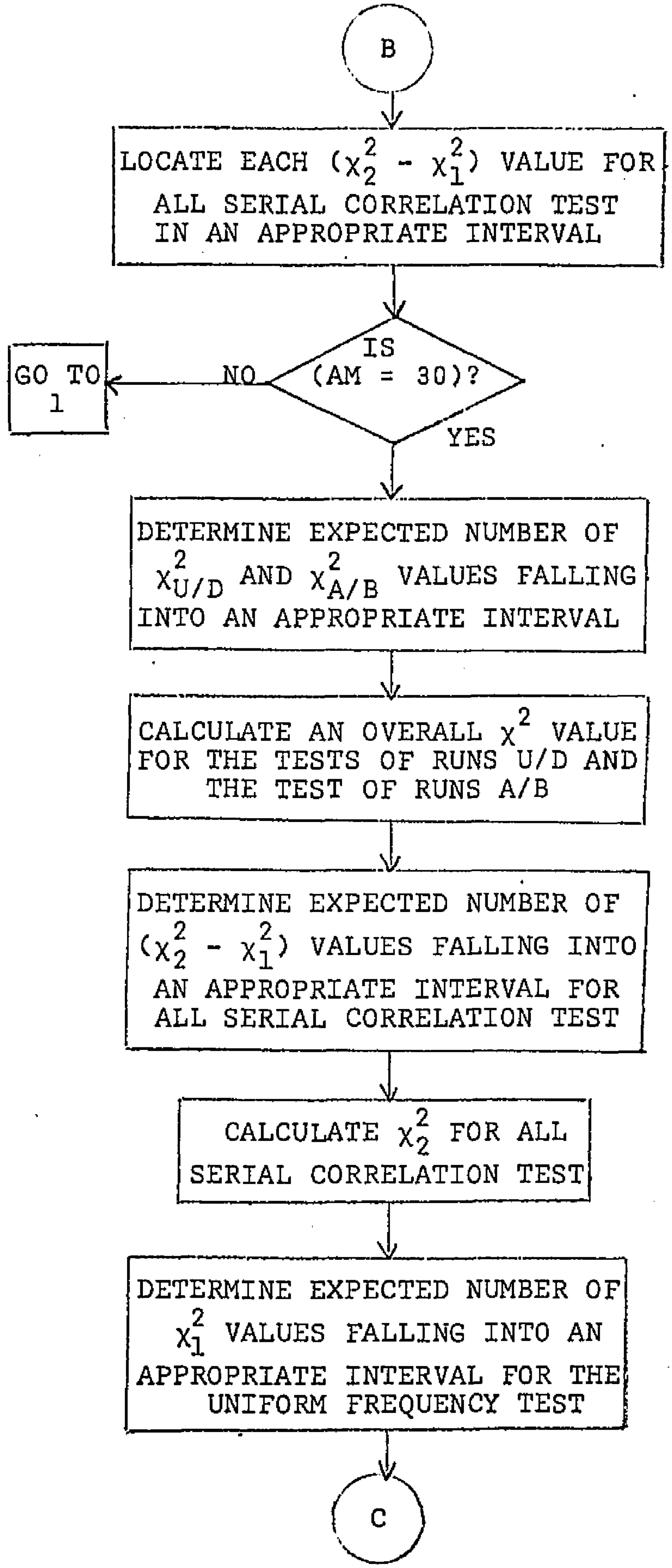
RANDOM NUMBER GENERATOR

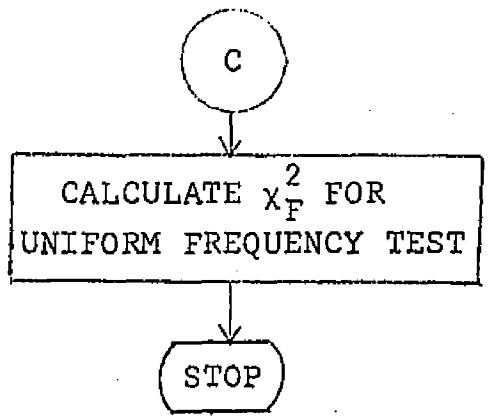


FLOW CHART OF THE STATISTICAL TEST RUN ON THE RANDOM NUMBER GENERATOR




A
DETERMINE EXPECTED NUMBER OF RUNS OF VARIOUS SIZES FOR RUNS TEST A/B
CALCULATE $\chi^2_{A/B}$ FOR RUNS TEST A/B
LOCATE $\chi^2_{U/D}$ VALUE FOR RUNS TEST U/D IN AN APPROPRIATE INTERVAL
LOCATE $\chi^2_{A/B}$ VALUE FOR RUNS TEST A/B IN AN APPROPRIATE INTERVAL
CALCULATE χ^2_1 VALUE FOR UNIFORM FREQUENCY TEST
LOCATE χ^2_1 VALUE FOR UNIFORM FREQUENCY TEST IN AN APPROPRIATE INTERVAL
CALCULATE EACH χ^2_2 VALUE FOR ALL SERIAL CORRELATION TEST
DETERMINE EACH $(\chi^2_2 - \chi^2_1)$ VALUE FOR ALL SERIAL CORRELATION TEST
B


B
LOCATE EACH $(\chi_2^2 - \chi_1^2)$ VALUE FOR ALL SERIAL CORRELATION TEST IN AN APPROPRIATE INTERVAL
IS (AM = 30)?
NO
GO TO 1
YES
DETERMINE EXPECTED NUMBER OF $\chi^2_{U/D}$ AND $\chi^2_{A/B}$ VALUES FALLING INTO AN APPROPRIATE INTERVAL
CALCULATE AN OVERALL χ^2 VALUE FOR THE TESTS OF RUNS U/D AND THE TEST OF RUNS A/B
DETERMINE EXPECTED NUMBER OF $(\chi_2^2 - \chi_1^2)$ VALUES FALLING INTO AN APPROPRIATE INTERVAL FOR ALL SERIAL CORRELATION TEST
CALCULATE χ_2^2 FOR ALL SERIAL CORRELATION TEST
DETERMINE EXPECTED NUMBER OF χ_1^2 VALUES FALLING INTO AN APPROPRIATE INTERVAL FOR THE UNIFORM FREQUENCY TEST
C


C
CALCULATE χ_F^2 FOR
UNIFORM FREQUENCY TEST
STOP

FORTRAN PROGRAM OF THE RANDOM NUMBER GENERATOR

```
C       THE RANDOM NUMBER GENERATOR USED IS A TWO STAGE
C       MULTIPLICATIVE CONGRUENTIAL GENERATOR. THE FIRST STAGE
C       DETERMINES RANDOMLY THE IX(I) VALUE TO BE USED IN THE
C       DETERMINATION OF THE RANDOM NUMBER. THE SECOND STAGE
C       DETERMINES THE RANDOM NUMBER ITSELF. THIS PROCEDURE IS
C       USED TO MINIMIZE THE SERIAL CORRELATION. IX(I) IS THE
C       POSITION OF THE 11 INITIAL VALUES. THE LAST VALUE IE.
C       IX(11) IS THE INITIAL VALUE FOR THE INDEX GENERATOR.
C       THE REMAINING 10 INITIAL VALUES ARE THE VALUES TO BE
C       USED IN DETERMINING THE RANDOM NUMBERS THEMSELVES.
        DIMENSION IX(11)
        COUNT=0.0
        DO2 I=1,11
        IX(I)=0
      2 CONTINUE
        DO3I=1,11
        READ(5,90) IX(I)
     90 FORMAT(I7)
      3 CONTINUE
      4 COUNT=COUNT+1.
        I=11
     20 IIX=IX(I)
        IY=IIX* 46331
        IF(IY)5,6,6
      5 IY=IY+2147483647 +1
      6 YFL=IY
        IX(I)=IY
        YFL=YFL*.4656613E-9
        IF(I.NE.11)GO TO 21
        IF(YFL.LT..1) GO TO 11
        IF(YFL.LT..2) GO TO 12
        IF(YFL.LT..3) GO TO 13
        IF(YFL.LT..4) GO TO 14
        IF(YFL.LT..5) GO TO 15
        IF(YFL.LT..6) GO TO 16
        IF(YFL.LT..7) GO TO 17
        IF(YFL.LT..8) GO TO 18
        IF(YFL.LT..9) GO TO 19
        I=10
        GO TO 20
     11 I=1
        GO TO 20
     12 I=2
        GO TO 20
     13 I=3
        GO TO 20
     14 I=4
        GO TO 20
     15 I=5
        GO TO 20
     16 I=6
```

```
      GO TO 20
   17 I=7
      GO TO 20
   18 I=8
      GO TO 20
   19 I=9
      GO TO 20
   21 WRITE(6,91) YFL
   91 FORMAT(/1X,'RANDOM NUMBER IS',4X,F20.15)
      IF(COUNT.LT. 100) GO TO 4
      DO7I=1,11
      WRITE(6,92)I,IX(I)
   92 FORMAT(/1X,'IX(',I3,') IS',2X,I10)
    7 CONTINUE
      STOP
      END

C              THE ELEVEN INITIAL VALUES (IX) USED

C               WITH THE RANDOM NUMBER GENERATOR

C                       IX(I)        NUMBER
C                         1          5873153
C                         2          8245319
C                         3          1137861
C                         4          7316141
C                         5          1860819
C                         6          2562763
C                         7          3534033
C                         8          1678567
C                         9          1448453
C                        10          1074817
C                        11          1635359
```

```
C           COMPUTER PROGRAM OF THE STATISTICAL TEST
C
C              RUN ON THE RANDOM NUMBER GENERATOR
C
C           THIS PROGRAM CONTAINS A MULTIPLICATIVE
C       CONGRUENTIAL RANDOM NUMBER GENERATOR AND SIX DIFFERENT
C       TEST THAT CAN BE USED TO TEST ANY PSEUDORANDOM NUMBER
C       GENERATOR. THE TEST INCLUDE (1) A UNIFORM FREQUENCY
C       TEST, (2), (3), (4) TEST FOR FIRST, SECOND, AND THIRD
C       ORDER SERIAL CORRELATION, (5) A TEST FOR RUNS ABOVE
C       AND BELOW THE MEAN, AND (6) A TEST FOR RUNS UP AND
C       DOWN. THESE TEST ARE DESIGNED SO THAT THEY CAN BE
C       AFFIXED TO ANY PROGRAM WHICH WILL GENERATE PSEUDO-
C       RANDOM NUMBERS. IN ADDITION THE SERIAL CORRELATION
C       TESTING PROCEDURE CAN BE MODIFIED SO AS TO TEST FOR
C       HIGHER ORDER SERIAL CORRELATION.
C           THE RANDOM NUMBERS ARE GENERATED IN 'AM' SETS OF
C       'AN' NUMBERS. FOR EACH SET OF 'AN' NUMBERS ALL SIX
C       TEST ARE CONDUCTED. THE RESULTS FOR EACH TEST IN TURN
C       ARE THEN COMBINED INTO A CORRESPONDING TEST ON ALL
C       'AN' NUMBERS.
C
C       A LIST OF THE MAJOR VARIABLES FOLLOWS.
C       ***************************************************
C
C                VARIABLES IN THE RANDOM NUMBER GENERATOR
C       IX(I).....INITIAL VALUES. THESE ARE READ IN.
C       AN........NUMBER OF RANDOM NUMBERS IN A GROUP (GE. 50)
C       AM........NUMBER OF GROUPS (GE. 20)
C       YFL.......RANDOM NUMBER IN FLOATING POINT MODE
C       IY........RANDOM NUMBER IN FIXED POINT MODE. THE NEW
C                 IX(I).
C
C                VARIABLES IN THE FREQUENCY TEST
C       XINTR.....NUMBER OF CLASSES INTO WHICH A RANDOM NUMBER
C                 IS PLACED.
C       U.........NUMBER OF SUBDIVISIONS INTO WHICH THE AM
C                 DIFFERENT CHI-SQUARE VALUES ARE PLACED FOR
C                 THE OVER-ALL CHI-SQUARE TEST.
C       CHISQ.....THE CHI-SQUARE VALUE FOR THE UNIFORMITY
C                 TEST FOR EACH SET AM.
C       FCHISQ....THE CHI-SQUARE VALUE FOR THE UNIFORMITY
C                 TEST FOR ALL AM SETS.
C
C                VARIABLES IN THE CORRELATION TEST
C       IRHO......INDICATES HIGHEST ORDER SERIAL TEST TO BE
C                 PERFORMED.
C       TCHISQ....THE CHI-SQUARE VALUE FOR THE SERIAL TEST.
C                 FIRST ORDER TEST IS SUBSCRIPTED 1, SECOND
C                 ORDER TEST IS SUBSCRIPTED 2, THIRD ORDER
C                 TEST IS SUBSCRIPTED 3.
C       DICHI.....THE DIFFERENCE BETWEEN FCHISQ AND TCHISQ.
C                 THIS VALUE IS CHI-SQUARE DISTRIBUTED AND IS
```

```
C               USED IN TESTING EACH SET AM FOR 1, 2, AND 3
C               ORDER SERIAL CORRELATION.
C     SCHI......THE CHI-SQUARE VARIABLE USED TO TEST ALL AM
C               SETS FOR 1, 2, AND 3 ORDER SERIAL
C               CORRELATION.
C
C              VARIABLES IN RUNS TESTS
C       RUNS ABOVE AND BELOW THE MEAN
C     IRUN......NUMBER OF RUNS OF THE SIZE INDICATED IN THE
C               SUBSCRIPT.
C     ICOUNT....NUMBER OF TOTAL RUNS.
C     CHIRUN....CHI-SQUARE VALUE COMPARING ACTUAL RUN
C               SIZES WITH EXPECTED RUN SIZES.
C     CHIRUT....CHI-SQUARE VALUE USED TO TEST ALL AM SETS
C               FOR RUN SIZES ABOVE AND BELOW THE MEAN.
C       RUNS UP AND DOWN
C     JRUN......ALL THESE VARIABLES CORRESPOND TO THEIR
C     JCOUNT    COUNTERPARTS IN THE RUNS TEST DESCRIBED
C     CHIRUN    ABOVE.
C     CHIRUT
C
C
C     DIMENSION STATEMENTS
C     ***************************************************************
      DIMENSION IX(11),FJK(3,10,10)
      DIMENSION FJ(10),CAPFJ(4),DICHI(3),SUM(3),DIFFJK(3)
      DIMENSION SDIFF(3),S(3,4),SUDI(3),SCHI(3),TCHISQ(3)
      DIMENSION IRUN(20),JRUN(20),EXRU(20),CHIRUN(2)
      DIMENSION CHIRUT(2,4),SDI(2),DI(2),RUCHI(2)
C     INITIALIZING STATEMENTS FOR THE 'AM' SETS
C     ***************************************************************
      AN=400.
      N=AN
      AM=30.
      M=AM
      XINTR=10.
      U=4.0
      IRHO=3
      IIU=U
      INTR=XINTR
      DO2 I=1,11
      IX(I)=0
    2 CONTINUE
      DO3I=1,11
      READ(5,90) IX(I)
   90 FORMAT(I10)
    3 CONTINUE
      DO2000IQQ=1,12
      DO62I=1,IIU
      CAPFJ(I)=0.0
   62 CONTINUE
      DO85I=1,IRHO
      DO85J=1,IIU
      S(I,J)=0.
```

```
   85 CONTINUE
      DO84I=1,2
      DO84J=1,IIU
      CHIRUT(I,J)=0.
   84 CONTINUE
C     INITIALIZING STATEMENTS FOR EACH SET IN TURN
C     ***************************************************************
      KK=0
   86 KK=KK+1
      DO83I=1,2
      CHIRUN(I)=0.
   83 CONTINUE
      DO29I=1,INTR
      FJ(I)=0.
   29 CONTINUE
      DO28IS=1,IRHO
      DO 28LJ=1,INTR
      DO28LK=1,INTR
      FJK(IS,LJ,LK)=0.
   28 CONTINUE
      LJ=0
      LJK=0
      LKJ=0
      LK=0
      IGR=0
      ILT=0
      ICOUNT=0
      DO107I=1,20
      IRUN(I)=0
  107 CONTINUE
      JCOUNT=0
      IUP=0
      IDW=0
      DO140I=1,20
      JRUN(I)=0
  140 CONTINUE
      DO50II=1,N
C     THE RANDOM NUMBER GENERATOR.
C     A MULTIPLICATIVE CONGRUENTIAL GENERATOR WITH MULTIPLE
C     STARTING POINTS ***************************************
      I=11
   20 IIX=IX(I)
      IY=IIX* 46331
      IF(IY)5,6,6
    5 IY=IY+2147483647 +1
    6 YFL=IY
      IX(I)=IY
      YFL=YFL*.4656613E-9
      IF(I.NE.11)GO TO 21
      IF(YFL.LT..1) GO TO 11
      IF(YFL.LT..2) GO TO 12
      IF(YFL.LT..3) GO TO 13
      IF(YFL.LT..4) GO TO 14
      IF(YFL.LT..5) GO TO 15
```

```
      IF(YFL.LT..6) GO TO 16
      IF(YFL.LT..7) GO TO 17
      IF(YFL.LT..8) GO TO 18
      IF(YFL.LT..9) GO TO 19
      I=10
      GO TO 20
   11 I=1
      GO TO 20
   12 I=2
      GO TO 20
   13 I=3
      GO TO 20
   14 I=4
      GO TO 20
   15 I=5
      GO TO 20
   16 I=6
      GO TO 20
   17 I=7
      GO TO 20
   18 I=8
      GO TO 20
   19 I=9
      GO TO 20
C     DETERMINING RUN SIZES FOR RUNS (UP &DOWN) TEST
C     *************************************************************
   21 IF(II.EQ.1) GO TO 129
      IF(FYFL.GT.YFL) GO TO 121
      IF(IUP.EQ.0) GO TO 120
      IUP=IUP+1
      IF(II.EQ.N) JRUN(IUP)=JRUN(IUP)+1
      GO TO 129
  120 JCOUNT=JCOUNT+1
      IF(II.GT.2)JRUN(IDW)=JRUN(IDW)+1
      IDW=0
      IUP=IUP+1
      IF(II.EQ.N) JRUN(IUP)=JRUN(IUP)+1
      GO TO 129
  121 IF(IDW.EQ.0) GO TO 123
      IDW=IDW+1
      IF(II.EQ.N) JRUN(IDW)=JRUN(IDW)+1
      GO TO 129
  123 JCOUNT=JCOUNT+1
      IF(II.GT.2) JRUN(IUP)=JRUN(IUP)+1
  124 IUP=0
      IDW=IDW+1
      IF(II.EQ.N) JRUN(IDW)=JRUN(IDW)+1
  129 FYFL=YFL
C     DETERMINING RUN SIZES FOR RUNS (ABOVE& BELOW) TEST
C     *************************************************************
      IF(YFL.GT..5) GO TO 102
      IF(ILT.EQ.0) GO TO 101
      ILT=ILT+1
      IF(II.EQ.N) IRUN(ILT)=IRUN(ILT)+1
```

```
      GO TO 105
  101 ICOUNT=ICOUNT+1
      IF(II.EQ.1) GO TO 104
      IRUN(IGR)=IRUN(IGR)+1
  104 IGR=0
      ILT=ILT+1
      IF(II.EQ.N) IRUN(ILT)=IRUN(ILT)+1
      GO TO 105
  102 IF(IGR.EQ.0) GO TO 103
      IGR=IGR+1
      IF(II.EQ.N) IRUN(IGR)=IRUN(IGR)+1
      GO TO 105
  103 ICOUNT=ICOUNT+1
      IF(II.EQ.1) GO TO 106
      IRUN(ILT)=IRUN(ILT)+1
  106 ILT=0
      IGR=IGR+1
      IF(II.EQ.N) IRUN(IGR)=IRUN(IGR)+1
C     DETERMINING THE INTERVALS INTO WHICH THE RANDOM
C     NUMBERS FALL FOR THE UNIFORM FREQUENCY TEST AND THE
C     1ST,2ND,3RD ORDER SERIAL CORRELATION TEST
C     ******************************************************************
  105 IF(YFL    .LT. .1) GO TO 41
      IF(YFL    .LT. .2) GO TO 42
      IF(YFL    .LT. .3) GO TO 43
      IF(YFL    .LT. .4) GO TO 44
      IF(YFL    .LT. .5) GO TO 45
      IF(YFL    .LT. .6) GO TO 46
      IF(YFL    .LT. .7) GO TO 47
      IF(YFL    .LT. .8) GO TO 48
      IF(YFL    .LT. .9) GO TO 49
      L=10
      GO TO 51
   41 L=1
      GO TO 51
   42 L=2
      GO TO 51
   43 L=3
      GO TO 51
   44 L=4
      GO TO 51
   45 L=5
      GO TO 51
   46 L=6
      GO TO 51
   47 L=7
      GO TO 51
   48 L=8
      GO TO 51
   49 L=9
   51 FJ(L)=FJ(L)+1.
      IF(II.EQ.1) GO TO 75
      IF(II-3) 70,71,72
   70 LJK=L
```

```
      GO TO 50
   71 LKJ=L
      GO TO 50
   72 LK=L
      FJK(1,LJ,LJK)=FJK(1,LJ,LJK)+1.
      FJK(2,LJ,LKJ)=FJK(2,LJ,LKJ)+1.
      FJK(3,LJ,LK )=FJK(3,LJ,LK )+1.
      LJ=LJK
      LJK=LKJ
      LKJ=LK
      IF(II.NE.N) GO TO 50
      FJK(1,LJ,LJK)=FJK(1,LJ,LJK)+1.
      FJK(1,LJK,LKJ)=FJK(1,LJK,LKJ)+1.
      FJK(2,LJ,LKJ)=FJK(2,LJ,LKJ)+1.
      GO TO 50
   75 LJ=L
   50 CONTINUE
C     CHI-SQUARE TEST ON TOTAL RUNS AND RUN SIZES FOR RUN UP
C     UP AND DOWN ****************************************
      CNT=JCOUNT
      EXCNT=(2.*AN-1.)/3.
      CHITER=((CNT-EXCNT)**2)/EXCNT
      SUEXRU=0.
      DO130I=1,N
      AI=I
      FAC=AI+3.
      R=FAC
      DO131J=1,N
      AJ=J
      IF((R-AJ).EQ.1.) GO TO 132
      FAC=FAC*(R-AJ)
  131 CONTINUE
  132 EXRU(I)=2.*(((AI**2)+3.*AI+1.)*AN-((AI**3)+3.*(AI**2)-
     1AI-4.))/FAC
      SUEXRU=SUEXRU+EXRU(I)
      IF(EXRU(I).GE.5.) GO TO 130
      EXRU(I)=((2.*AN-1.)/3.)-(SUEXRU-EXRU(I))
      IF(EXRU(I).GE.5.) GO TO 160
      EXRU(I-1)=EXRU(I-1)+EXRU(I)
      IRJ=I-1
      GO TO161
  160 IRJ=I
      GO TO 161
  130 CONTINUE
  161 JDUM=0
      DO170I=1,IRJ
      IF(I.EQ.IRJ)GO TO 170
      JDUM=JDUM+JRUN(I)
  170 CONTINUE
      JRUN(IRJ)=JCOUNT-JDUM
      DO162I=1, IRJ
      RUDIFF=((JRUN(I)-EXRU(I))**2)/EXRU(I)
      II=1
      CHIRUN(II)=CHIRUN(II)+RUDIFF
```

```
  162 CONTINUE
      CHIRUN(II)=CHIRUN(II)+CHITER
C     CHI-SQUARE TEST ON TOTAL RUNS AND RUN SIZES FOR RUN
C     ABOVE AND BELOW ************************************
      CNT=ICOUNT
      EXCNT=(AN+1.)/2.
      CHITER=((CNT-EXCNT)**2)/EXCNT
      SUEXRU=0.
      DO110I=1,N
      AI=I
      EXRU(I)=(AN-AI+3.)/(2.**(I+1))
      SUEXRU=SUEXRU+EXRU(I)
      IF(EXRU(I).GE.5.) GO TO 110
      EXRU(I)=((AN+1.)/2.)-(SUEXRU-EXRU(I))
      IF(EXRU(I).GE.5.) GO TO 164
      EXRU(I-1)=EXRU(I-1)+EXRU(I)
      IRI=I-1
      GO TO 165
  164 IRI=I
      GO TO165
  110 CONTINUE
  165 JDUM=0
      DO171I=1,IRI
      IF(I.EQ.IRI)GO TO 171
      JDUM=JDUM+IRUN(I)
  171 CONTINUE
      IRUN(IRI)=ICOUNT-JDUM
      DO166I=1,IRI
      RUDIFF=((IRUN(I)-EXRU(I))**2)/EXRU(I)
      II=2
      CHIRUN(II)=CHIRUN(II)+RUDIFF
  166 CONTINUE
      CHIRUN(II)=CHIRUN(II)+CHITER
C     POSITIONING CHI-SQUARE VALUES OF RUNS TEST INTO
C     APPROPRIATE INTERVALS ******************************
      DO150I=1,2
      IF(I.EQ.2) GO TO 167
      IF(CHIRUN(I)   .LT.1.92256      )GO TO 151
      IF(CHIRUN(I)   .LT.3.35669      )GO TO 152
      IF(CHIRUN(I)   .LT.5.38527      )GO TO 153
      GO TO 168
  167 IF(CHIRUN(I)   .LT.3.45460      )GO TO 151
      IF(CHIRUN(I)   .LT.5.34812      )GO TO 152
      IF(CHIRUN(I)   .LT.7.84080      )GO TO 153
  168 IU=4
      GO TO 154
  151 IU=1
      GO TO 154
  152 IU=2
      GO TO 154
  153 IU=3
  154 CHIRUT(I,IU)=CHIRUT(I,IU)+1.
  150 CONTINUE
C     CALCULATING CHI-SQUARE(1) VALUES
```

```
C     ******************************************************************
      SUDIFF=0.
      DO53 J=1,INTR
      DIFF=(FJ(J)-AN/XINTR)**2
      SUDIFF=SUDIFF+DIFF
   53 CONTINUE
      CHISQ=(XINTR/AN)*SUDIFF
C     POSITIONING EACH CHI-SQUARE(1) VALUE INTO INTERVALS
C     ******************************************************************
      IF(CHISQ     .LT.5.89883) GO TO 63
      IF(CHISQ     .LT.8.34283) GO TO 64
      IF(CHISQ     .LT.11.3888) GO TO 65
      IU=4
      GO TO 36
   63 IU=1
      GO TO 36
   64 IU=2
      GO TO 36
   65 IU=3
   36 CAPFJ(IU)=CAPFJ(IU)+1.
C     CALCULATING CHI-SQUARE(2) AND CHI-SQUARE(2) MINUS
C     CHI-SQUARE(1)   ****************************************
      DO77IS=1,IRHO
      SUM(IS)=0.
   77 CONTINUE
      DO76IS=1,IRHO
      DO32LJ=1,INTR
      DO32LK=1,INTR
      DIFFJK(IS)=(FJK(IS,LJ,LK)-(AN-IS)/XINTR**2)**2
      SUM(IS)=SUM(IS)+DIFFJK(IS)
   32 CONTINUE
      TCHISQ(IS)=((XINTR**2)/(AN-IS))*SUM(IS)
      DICHI(IS)=TCHISQ(IS)-CHISQ
C     POSITIONING EACH CHI-SQUARE(2) MINUS CHI-SQUARE(1)
C     VALUE INTO INTERVALS ***********************************
      IF(DICHI(IS)    .LT.80.6247     )GO TO 33
      IF(DICHI(IS)    .LT.89.3342     )GO TO 34
      IF(DICHI(IS)    .LT.98.6499     )GO TO 35
      IU=4
      GO TO 67
   33 IU=1
      GO TO 67
   34 IU=2
      GO TO 67
   35 IU=3
   67 S(IS,IU)=S(IS,IU)+1.
   76 CONTINUE
   68 IF(KK.LT.M) GO TO 86
C     CALCULATING OVERALL CHI-SQUARE VALUE ON RUN SIZES FOR
C     ALL SETS ***********************************************
      DO157I=1,2
      SDI(I)=0.
  157 CONTINUE
      DO156I=1,2
```

```
      DO155J=1,IIU
      DI(I)=(CHIRUT(I,J)-AM/U)**2
      SDI(I)=SDI(I)+DI(I)
  155 CONTINUE
      RUCHI(I)=(U/AM)*SDI(I)
  156 CONTINUE
C     CALCULATING OVERALL CHI-SQUARE VALUE ON 1ST, 2ND, AND
C     3RD ORDER SERIAL CORRELATION *************************
      DO82IS=1,IRHO
      SUDI(IS)=0.
   82 CONTINUE
      DO81IS=1,IRHO
      DO37J=1,IIU
      SDIFF(IS)=(S(IS,J)-AM/U)**2
      SUDI(IS)=SUDI(IS)+SDIFF(IS)
   37 CONTINUE
      SCHI(IS)=(U/AM)*SUDI(IS)
   81 CONTINUE
C     CALCULATING OVERALL CHI-SQUARE VALUE ON UNIFORM
C     FREQUENCY TEST ************************************
      SUCADF=0.0
      DO69J=1,IIU
      CADIF=(CAPFJ(J)-AM/U)**2
      SUCADF=SUCADF+CADIF
   69 CONTINUE
      FCHISQ=(U/AM)*SUCADF
      IF(IQQ.NE.1) GO TO 1111
      WRITE(6,1000)
 1000 FORMAT(/47X,'TABLE 1')
      WRITE(6,1001)
 1001 FORMAT(/31X,'TEST RESULTS ON RANDOM NUMBER GENERATOR')
      WRITE(6,1002)
 1002 FORMAT(21X,'
     1                    ')
      WRITE(6,1002)
      WRITE(6,1003)
 1003 FORMAT(/22X,'SEQUENCE',3X,'UNIFOR-',7X,'CORRELATION',
     112X,'RUNS')
      WRITE(6,1004)
 1004 FORMAT(23X,'NUMBER',5X,'MITY',2X,'
     1                         ')
      WRITE(6,1005)
 1005 FORMAT(/42X,'FIRST',2X,'SECOND',3X,'THIRD',2X,'ABOVE-'
     1,4X,'UP-')
      WRITE(6,1006)
 1006 FORMAT(42X,'ORDER',3X,'ORDER',3X,'ORDER',2X,'BELOW',4X
     1,'DOWN')
      WRITE(6,1002)
 1111 WRITE(6,1007)FCHISQ,SCHI(1),SCHI(2),SCHI(3),RUCHI(2),
     1RUCHI(1)
 1007 FORMAT(//33X,6(F6.4,2X))
      IF(IQQ.NE.12) GO TO 2000
      WRITE(6,1002)
      WRITE(6,1008)
```

```
1008 FORMAT(/28X,'THE CRITICAL VALUE OF CHI-SQUARE ARE')
     WRITE(6,1009)
1009 FORMAT(57X,'FOR    = .05 IS 7.81473')
     WRITE(6,1010)
1010 FORMAT(57X,'FOR    = .01 IS 11.3449')
2000 CONTINUE
     STOP
     END
```

TABLE 1

TEST RESULTS ON RANDOM NUMBER GENERATOR

SEQUENCE NUMBER	UNIFORMITY	CORRELATION			RUNS	
		FIRST ORDER	SECOND ORDER	THIRD ORDER	ABOVE-BELOW	UP-DOWN
1 - 12000	5.4667	2.2667	2.5333	0.6667	0.6667	6.8000
12001 - 24000	2.8000	0.6667	1.2000	3.6000	0.4000	6.0000
24001 - 36000	1.7333	1.4667	2.2667	3.3333	2.2667	1.2000
36001 - 48000	4.6667	2.2667	4.9333	2.2667	4.9333	6.5333
48001 - 60000	6.0000	3.6000	2.2667	5.7333	2.5333	1.4667
60001 - 72000	2.2667	2.5333	7.0667	2.8000	2.5333	4.6667
72001 - 84000	6.0000	0.6667	1.2000	5.4667	0.1333	3.6000
84001 - 96000	0.6667	0.1333	2.8000	3.6000	1.4667	3.6000
96001 -108000	1.2000	4.4000	1.2000	3.8667	2.8000	2.5333
108001 -120000	2.2667	6.8000	8.6667	0.1333	5.7333	2.2667
120001 -132000	0.6667	8.1333	4.9333	2.2667	2.5333	6.8000
132001 -144000	0.6667	2.8000	0.6667	1.2000	9.2000	0.4000

THE CRITICAL VALUE OF CHI-SQUARE
FOR α = .05 IS 7.81473
FOR α = .01 IS 11.3449

CHAPTER IV

A STATEMENT OF THE EXPERIMENTAL PROCEDURE

Introduction

The simulation model constructed for this study included three alternative deterministic equivalents to the stochastic programming model. These deterministic equivalents were (1) the two-stage slack approach to programming under uncertainty, (2) the active approach to stochastic linear programming under risk, and (3) the one-stage expected value approach. The experimental model evaluated an empirical linear programming problem with stochastic parameters in terms of each deterministic equivalent stated above. A Monte Carlo simulation of this empirical problem was then performed. The results of this simulation were used as a standard with which to evaluate the results of each deterministic equivalent.

The objective of this chapter is to specify the characteristics of the experimental model. Initially the specific empirical problem used to generate the experimental results is stated. Next, the experimental design is reviewed. Emphasis is placed upon the selection of the various initial stochastic formulations of the empirical problem that were analyzed by the experimental model. The three deterministic equivalents are then presented. Special attention here is placed upon the procedure used to include these deterministic equivalents in the experimental model. The chapter concludes with an appendix which contains the flow chart for computation of each deterministic equivalent and the flow chart and FORTRAN program of the entire experimental model.

The Problem Used in the Experimental Model

The empirical problem used in this study to generate the experimental results is a modification of an agricultural production problem which has appeared in the literature concerning stochastic linear programming.[1] The problem, as it appears in the literature, contains five variables and five constraints. Because of the numerous ways in which uncertainty can be introduced into the parameters of the problem, the dimensions of the problem are reduced for present purposes. The modified problem contains three variables and three constraints. With a (3 x 3) problem there are sixty-three initial formulations of the problem which can be developed, in terms of combinations of parameters which are taken to be stochastic.

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[1] M. M. Babbar, "Distributions of Solutions of a Set of Linear Equations (With an Application to Linear Programming)," Journal of the American Statistical Association, L (September, 1955), 854-869; and J. K. Sengupta and J. H. Partillo-Campbell, "A Fractile Approach to Linear Programming Under Risk," Management Science, XVI (January, 1970), 298-308.

The empirical problem used in this study can be stated as

$$\begin{aligned} \text{Maximize:} \quad & Z = C'X \\ \text{subject to:} \quad & AX \leqq B \quad \text{and} \\ & X \geqq 0 \quad , \end{aligned} \qquad [1]$$

where the expected values of the parameters in [1] are:

$$A = \begin{bmatrix} 0.31772 & 0.96956 & 0.27870 \\ 0.02274 & 0.92490 & 0.02770 \\ 0.02555 & 0.21186 & 0.07523 \end{bmatrix} ,$$

$$B = \begin{bmatrix} 1800 \\ 148 \\ 234 \end{bmatrix} , \text{ and } \quad C = \begin{bmatrix} 1.56 \\ 3.81 \\ 0.84 \end{bmatrix} . \qquad [2]$$

The variables of the problem are the quantities (in bushels) of the agricultural products corn, flax, and oats which are to be produced subject to the available resources of capital, land, and labor. These resources are respectively expressed in dollars, acres, and man-hours.

This particular (3 x 3) problem was selected from the original (5 x 5) problem after an evaluation of the optimum solutions of the various (3 x 3) problems that can be formed from the original problem. In [2] above the C vector contains the expected values of the profit margin of the more important variables from the original problem, while the B vector contains the expected values of the more important resources available. The A matrix contains the technological coefficients relating the amount of any particular resource required to produce any product. These values were assumed to be constant throughout the entire experiment.

Significance of the expected value solution

In addition to being one of the deterministic equivalents evaluated, the expected value solution was used in a number of ways in the experimental model. Initially the expected value solution was used to aid in the design of the experiment. The optimum solution vector of the problem described in [1] and [2], when all the parameters are assumed constant and equal to their expected values, contains non-zero values for the variables X_1, X_5, X_6. The variable X_1 refers to the first product, while the variables X_5 and X_6 are slack variables found in the second and third constraints respectively. The amount of slack, X_5, in the second constraint was found to be proportionally less than the amount of slack, X_6, in the third constraint.

Based on this expected value solution, it was decided that X_1 is the most important variable in the problem since it is the only non-zero decision variable which appears in the optimum solution vector of the expected value model. Since the first constraint is the only constraint which has no slack in the optimum solution, then b_1 was viewed as the most important resource in the problem. Of the remaining variables X_2 has the larger profit margin. This factor placed it as the next most important variable in the problem. Correspondingly, b_2 was the next most important resource since the second constraint has relatively less slack than the third constraint.

The ranking of the variables and the resources of the problem was important when one considers that not all the initial formulations of the problem were to be analyzed. There are seven ways in which uncertainty can be introduced into either the b or the c vectors and forty-nine ways in which uncertainty can be introduced into both vectors simultaneously. This is a total of sixty-three initial formulations that can be analyzed. Each formulation may also be considered under different assumptions concerning the standard deviation of the stochastic parameters which appear. Specifically in this study each initial formulation was analyzed under six different values of the standard deviation of the stochastic parameters. In view of this fact there are then 378 (63 x 6) initial formulations which can be analyzed.

The expected value solution, by providing a means for ranking the variables and the resources of the problem, was used to determine which of the 378 initial formulations are significant enough to be analyzed, and thus, to omit from analysis those judged to be insignificant. The significant formulations were considered to be those which contain as stochastic parameters either the amount available of the most significant resource or the profit margin of the most significant variable.

An alternative experimental problem

The experimental problem as it is stated in [1] and [2] is not a tightly constrained problem. This can be determined from a consideration of the optimum solution vector of the problem when the expected values of the parameters are used to determine a solution. The solution vector, X_{EV}, of the expected value approach indicates that 5665.4 units of X_1 are to be produced, that 19.2 units of slack are present in the second constraint, and that 89.2 units of slack are present in the third constraint.

It is highly probable, with these large amounts of slack in the last two constraints of the problem, that the optimum solutions generated assuming either a stochastic b_2 or b_3 are insensitive to the changing values of these parameters. Consider the third constraint where 89.2 units of slack are available. If the coefficient of variation of b_3 is as large as .30 then the standard deviation, σ_{b_3}, of b_3 is 70.2. A value of b_3 which would eliminate all the slack in this third constraint must be at least $1.27\sigma_{b_3}$ units less than the mean of b_3. Since b_3 is normally distributed, then the probability of generating a value of b_3 which would satisfy this condition is .1120. When the coefficient of variation of b_3

is smaller than .30, then this probability decreases. For small values of the coefficient of variation (say .05 or .10) it is highly improbable that this constraint would ever be a binding constraint. A similar argument can be presented concerning the second constraint, although the amount of slack in this case is much less and, therefore, the probability of generating a value of b_2 which would cause this constraint to be binding is somewhat greater.

Due to the slack present in the experimental problem stated in [1] and [2], an alteration of the problem is desirable so that less slack is present in the expected value solution. In order to make the first experimental problem a tightly constrained problem the following changes were made in the expected values of the parameters given in [2]. The profit margin, c_3, was increased to 1.50. This change made the production of X_3 more desirable and increased the possibility of X_3 entering the optimum solution vector of the expected value approach. In addition a_{12} was changed from 0.96956 to 0.66956 and a_{22} was changed from 0.9249 to 0.0549. In the first formulation of the experimental problem X_2 required the use of relatively more resources than was justified by its profit margin. These changes in the A matrix made the production of X_2 more desirable and caused this variable to enter the optimum solution vector which was derived from the expected value approach.

The optimum solution vector of the expected value approach to the modified experimental problem indicates that 4475.1 units of X_1 and 564.8 units of X_2 are to be produced. In addition there are 15.2 units of slack in the second constraint, while the first and the third constraints contain no slack units.

This modified experimental problem is more tightly constrained than the initial experimental problem. Two sets of experimental results were generated by the experimental model, one set for the slightly constrained initial problem and the other set for the more tightly constrained modified problem. These problems are referred to respectively as experimental problem A and experimental problem B.

The Experimental Design

The experimental procedure was divided into three phases. In the first phase only the B vector was considered to be stochastic, in the second phase only the C vector was stochastic, and in the third phase uncertainty was introduced into both vectors simultaneously. In the first and the second phases, where seven initial formulations are each possible, only four formulations each were considered; while in the third phase only nine of the forty-nine possible formulations were analyzed. This is a total of seventeen formulations which were analyzed under different conditions.

With regard to the first phase, the first formulation treated the most significant resource (b_1) as stochastic; the second formulation considered the two most significant resources (b_1, b_2) as stochastic; the third formulation assumed the most and the least significant resources (b_1, b_3) to be stochastic; and the fourth formulation treated all resources (b_1, b_2, b_3) as stochastic. The reader should observe that each formulation contains b_1 as a stochastic element. The other resource values, since they are of lesser importance, were included as stochastic parameters only in combination with a stochastic b_1.

Each of these four formulations was in turn analyzed under six different assumptions concerning the standard deviations of the stochastic parameters which appeared in the formulations. The four formulations described above, with the coefficient of variation (σ/μ) of each stochastic parameter equal to 0.5, comprised the first four experiments that were run. The next four experiments (numbers 5 to 8) combined the same four formulations with the coefficients of variation of the stochastic parameters equal to .10. To complete all the experiments in the first phase, the coefficients of variation of the stochastic parameters which appear were allowed, in turn, to equal .15, .20, .25, and .30. There were, then, twenty-four experiments (numbers 1 to 24) in this first phase.

The twenty-four experiments of the next phase (experiment numbers 25 to 48) were analogous to the corresponding set of experiments in the first phase. The first formulation in this phase treated the profit margin c_1 of the most significant variable as stochastic; the second formulation considered c_1 and c_2 as stochastic parameters, the third formulation considered c_1 and c_3 as stochastic parameters, and the fourth formulation treated all the profit margins c_1, c_2, and c_3 as stochastic. The coefficients of variation for each of the first four experiments (number 25 to 28) was .05. The coefficients of variation of the stochastic parameters were allowed, in turn, to equal .05, .10, .15, .20, .25, and .30, and each formulation described above was repeated six times.

When both the B and the C vectors contain stochastic parameters there are forty-nine formulations which can be analyzed. Nine representative formulations were selected to be included in this phase of the experiment. Again each of the nine formulations was analyzed with the six different values assumed by the coefficients of variation of the stochastic parameters which appeared in any formulation. In effect there were fifty-four experiments (numbers 49 to 102) included in phase three of the experiment. The parameters which were stochastic in these nine formulations are as follows.

Formulation	Stochastic Parameters
1	b_1, c_1
2	b_1, c_1, c_2
3	b_1, b_2, c_1
4	b_1, c_1, c_2, c_3
5	b_1, b_2, b_3, c_1
6	b_1, b_2, c_1, c_2
7	b_1, b_2, c_1, c_2, c_3
8	b_1, b_2, b_3, c_1, c_2
9	b_1, b_2, b_3, c_1, c_2, c_3 [3]

In the first nine experiments (numbers 49-57) of this phase the coefficients of variation of the stochastic parameters were .05. These nine formulations in [3] were repeated with the coefficient of variation varying up to .30 as was the case in the first two phases.

In summary this experiment was divided into three phases. The first phase and the second phase each contained four different formulations, while the third phase contained nine different formulations. Each of these seventeen formulations was analyzed under six different assumptions concerning the magnitude of the standard deviation of the stochastic parameters which appeared in the different formulations. This provided a total of 102 experiments which were analyzed by the simulation model.

The Experimental Procedure[2]

The simulation model is programmed in three parts. In the main program the definitions and assumptions pertaining to the study are presented; the input and output of data is controlled; and each deterministic equivalent is programmed. The first subroutine in the program (RAND) contains the pseudorandom number generator which has been adjusted to produce a normally distributed variable with a mean of zero and a standard deviation of one. The second subroutine (SIMPLX) is a simplex program which is designed to determine the optimum value of the objective function of any linear programming model. These different subroutines were called by the main program whenever their use was required.

The data pertaining to each of the experiments explained in the last section were read into the main program. For any particular experiment these data include (1) the number of stochastic parameters in the experiment, (2) the expected values of the stochastic parameters (this identifies which parameters are stochastic in the experiment), and (3) the coefficients of variation of each stochastic parameter.

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[2]In the discussion of the experimental procedure the variable names used in the FORTRAN program of the model are included where it would be beneficial to the reader to do so.

Each experiment was iterated 100 times. At the start of each iteration a value was determined for each stochastic parameter which appeared in the particular experiment. Since each stochastic parameter is assumed to be normally and independently distributed, these values were obtained by utilizing the subroutine RAND and the given mean and standard deviation of each of the parameters. The parameters in the model which were not assumed to be stochastic in any particular experiment were treated as constant at their expected values.

On each iteration, once the value of each parameter was determined, the problem was then adjusted, in turn, to conform to the specifications of the respective deterministic equivalents. The optimum value of the objective function of the problem was then determined, in turn, by utilizing each of the deterministic equivalents. The specific procedure involved in obtaining these optimum objective function values is explained in the remainder of this chapter.

The one-stage expected value approach (ZEXPC)

The one-stage expected value approach utilized in this experiment can be represented in the following way

$$\begin{aligned} &\text{Maximize:} && Z = E[C]X \\ &\text{subject to:} && AX \leq E[B] \quad \text{, and} \\ & && X \geq 0 \quad . \end{aligned} \qquad [4]$$

This deterministic equivalent replaces each stochastic parameter with its expected value. The optimum value of the objective function of the model in [4], assuming the expected values, is called ZEXPC. Since all the parameters were assumed to be constant with this approach, the ZEXPC value was constant for all the 102 experiments which were performed.

The simulation approach (ZSIM)

The simulation approach was next performed on each of the experimental formulations. Each formulation was iterated 100 times. The procedure involves: (1) a determination of which parameters are stochastic and the properties of those parameters; (2) for each stochastic parameter, subroutine RAND is used to determine a specific value of the parameter to use for each iteration; and (3) given the values of all the parameters as generated in (2), the subroutine SIMPLX is called and the conditional optimum value of the objective function is determined. The optimum value of the objective function on the ith iteration is $ZSIM_i$. Once all the iterations were completed, then the mean (ZSIMBR) and standard deviation (SDZSIM) of these optimum objective function values were computed.

The two-stage slack approach (ZTWS)

The two-stage slack approach used in this study can be represented in the following form

$$\begin{aligned} \text{Maximize:} \quad & C'X - E[F'Y] \\ \text{subject to:} \quad & AX + (Y^{+} - Y^{-}) \leq B, \\ & X, Y \geq 0 \end{aligned} \qquad [5]$$

where the second term in the objective function refers to the additional cost which may result from either excess slack or infeasibility in the final solution. In the set of constraints above when $B < AX$ then $Y^{+} = B - AX$ and $Y^{-} = 0$. This is the case of excess slack. The infeasible situation arises when $B < AX$, then $Y^{+} = 0$ and $Y^{-} = AX - B$.

This two-stage approach is based upon the premise that the investigator first solves the stochastic linear programming problem substituting an estimate for each of the stochastic parameters in the model. Once this initial solution is obtained, then the value of each of the stochastic parameters is observed and an adjustment made in the final solution reflecting any cost which may result from the development of either excess slack or infeasibility.

This procedure was reflected in the experimental model in the following way. The first stage solution was assumed to be the same as the expected value solution ZEXPC. It is assumed that the expected value of each stochastic parameter is the most reliable estimate the investigator has of that parameter. In accordance with this assumption the expected value solution is an appropriate initial solution. Each iteration of each experiment, therefore, began with ZEXPC as the optimum value of the objective function.

The adjustments made in the second stage depended upon which parameters were stochastic. In the FORTRAN program of the model this two-stage approach followed after the simulation approach. The stochastic parameters had been identified and a specific value for each had been obtained. The first step in the two-stage approach was to make an adjustment in ZEXPC if any of the parameters in the C vector were stochastic. This was required since ZEXPC had been determined by utilizing the expected values of the elements in the C vector. If some of the elements in the C vector were stochastic then the profit realized would depend upon the actual observed values of these stochastic elements. In the case where the stochastic elements were confined only to the B vector, then the profit realized, which was as yet unadjusted for any additional cost of excess slack or infeasibility, was the same as the expected profit ZEXPC.

Having adjusted ZEXPC for the effects produced by the stochastic elements in the C vector, the next step involved adjusting for the effects produced by any stochastic elements which may be present in the B vector. Only constraints which contained a stochastic b_i value had to be analyzed. If the solution vector of the expected value solution, X_{EV}, is premultiplied by A, the matrix of fixed technological coefficients, then this product AX_{EV} is a (3 x 1) vector which indicates the amount of the resources available which are accounted for in the expected value solution. In the quantity AX_{EV} an expected amount of slack associated with the different constraints is also accounted for. For each constraint which was stochastic the corresponding rows of AX_{EV} and B had to be compared. For example if the ith constraint is stochastic, then $\sum_{j=1}^{3} a_{ij}X_j$ must be compared with b_i, where b_i is the observed value of the stochastic parameter which was determined at the beginning of the current iteration. Infeasibility resulted when the summation above was greater than b_i while excess slack resulted when the summation was less than b_i. Excess slack for any stochastic constraint is defined as the slack in the constraint over and above the expected slack which was indicated in the expected value solution. In the case where both quantities were equal then there was no additional cost to be deleted from the ZEXPC value.

The cost of infeasibility and the cost of excess slack for the constraints are exogenous parameters which must be defined in the experimental model. The constraints of the problem used in this study assume the resources to be capital, land, and labor. In the case of excess slack the interpretation is that interest, rent, and wages were contracted for but were not required. When infeasibility results the interpretation is that additional amounts of these resources must be contracted for. In the latter case the unit costs of the different resources are assumed to be two times their regular unit costs.

The optimum value of the objective function determined with the two-stage slack approach is directly dependent upon the cost parameters which are used in the second stage of the approach to adjust the initial solution. This factor required an adjustment of the experimental model such that the two-stage slack approach was analyzed under alternative assumptions concerning the values of these cost parameters.

Three different sets of adjustment cost coefficients were used in the experimental model. The initial set of cost coefficients are

$$\text{COSLK} = \begin{pmatrix} .03 \\ 20.00 \\ 1.50 \end{pmatrix} \quad \text{and} \quad \text{COINF} = \begin{pmatrix} .06 \\ 40.00 \\ 3.00 \end{pmatrix} \quad . \qquad [6]$$

The second and third sets are obtained by multiplying the set in [6] first by two and then by four. The vector COSLK in [6] is the vector of cost coefficients relating to excess slack which may result in each of the constraints. The cost coefficients relating to infeasibility in the constraints are contained in the vector COINF.

The optimum value of the objective function for the two-stage approach is called ZTWS. As is the case with the simulation approach, each experiment was repeated 100 times with the mean (ZTWSBR) and the standard deviation (SDZTWS) of the optimum objective function values computed for each experiment.

<u>The active approach (ZACT)</u>

The active approach to stochastic linear programming is specifically designed to deal with stochastic elements which appear only in the constraints of the problem. Whenever stochastic elements appear only in the C vector, then the active approach must be applied to the dual problem which can be formed from the stochastic primal problem. In practice, when both the B and the C vectors contain stochastic elements, the active approach is applied to the problem after first substituting for each stochastic element in the C vector its expected value. If this procedure were carried out in the experimental model, then phase three of the experiment would be the same as phase one. Due to this fact, the active approach was only analyzed for the first two phases of the experiment.

The model of the active approach to linear programming under risk can be stated as

$$\begin{aligned} \text{Maximize:} \quad & Z = C'X \\ \text{subject to:} \quad & AX \leq BU \quad , \\ & X \geq 0 \end{aligned} \qquad [7]$$

where in the constraints X is a n-dimensional diagonal matrix, B is a m-dimensional diagonal matrix, and U is a (m x n) matrix. The allocation matrix U is such that u_{ij} is the proportion of the ith resource to be allocated to the production of the jth product. The active approach in this form assumes that all m constraints in the model contain stochastic elements. When some subset of the constraints of a problem contain stochastic elements, then the dimensions of the A, B, and U matrices are reduced. For example when r constraints contain stochastic elements, where $r < m$, then the dimensions of A are (r x n), the dimensions of B are(r x r), and the dimensions of U are (r x n). The (m - r) non-stochastic constraints are then added to the model in the same form as they would appear in a deterministic linear programming problem. For example in the (3 x 3) problem of this study if only b_1 is stochastic, then the constraints in the model of the active approach can be stated as

$$(a_{11}a_{12}a_{13}) \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{bmatrix} \leqq b_1(u_{11}u_{12}u_{13})$$

$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 \leqq b_2$$

$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 \leqq b_3$$

$$X_1, X_2, X_3 \geqq 0 \quad . \qquad [8]$$

This set of constraints can be reduced to

$$\begin{aligned}
&a_{11}X_1 && && && \leq b_1u_{11} \\
& && a_{12}X_2 && && \leq b_1u_{12} \\
& && && a_{13}X_3 && \leq b_1u_{13} \\
&a_{21}X_1 &+& a_{22}X_2 &+& a_{23}X_3 && \leq b_2 \\
&a_{31}X_1 &+& a_{32}X_2 &+& a_{33}X_3 && \leq b_3
\end{aligned} \qquad [9]$$

The number of constraints in the final formulation of the active approach increases as the number of stochastic constraints increases. When there are two stochastic constraints the final formulation contains seven constraints; while with three stochastic constraints there are nine constraints in the final formulation.

When the stochastic elements are limited to the C vector then the dual model can be stated as

$$\begin{aligned}
&\text{Minimize:} && Z = B'W \\
&\text{subject to:} && A'W \geq CV \quad , \\
& && W \geq 0
\end{aligned} \qquad [10]$$

where W is a m-dimensional diagonal matrix, C is a n-dimensional diagonal matrix, and V is a (n x m) matrix analogous to the allocation matrix U. The general statement of the active approach in [10] assumes that all n elements of C are stochastic. In the case where only some subset of the elements of the C vector are stochastic, then the dimensions of the matrices A', C, and V can be reduced. The procedure involved in arriving at the final formulation of the constraints in this dual case is analogous to the procedure described in the primal case. The only exception to the procedure is that in the dual model the sense of the constraints in the final formulation is "greater than or equal to" rather than "less than or equal to."

The programming of the active approach in the experimental model required that an assumption be made concerning the allocation ratios in U and V. These ratios are exogenous values which must be read into the program. The expected value solution to the experimental problem was again used to determine the values of these exogenous parameters. The expected value solution indicated that X_1 was the most significant variable in the problem. In fact X_1 is the only non-zero decision variable included in the optimum solution of the expected value approach. The importance of this variable can be reflected in the U matrix by placing increased weight upon the allocation ratios in the first column of the matrix. Even though increased weight is placed upon the allocation ratios related to the production of X_1, the production of the other products (X_2, X_3) must not be prohibited. The allocation ratios relating to all products must be greater than zero, so that the model allows for the possible production of all products.

Since the experimental results were dependent upon the specific values of the various allocation ratios which were used, the active approach was analyzed under two different sets of values for the allocation ratios of the U matrix. The allocation matrix U which was used in the first run of the experimental model is

$$U = \begin{bmatrix} .90 & .05 & .05 \\ .90 & .05 & .05 \\ .90 & .05 & .05 \end{bmatrix} . \qquad [11]$$

The ratios in this matrix are designed to allocate 90 per cent of each resource to the production of X_1, while at the same time allowing lesser amounts of the different resources for the production of the remaining products.

The allocation matrix used in the second run of the experimental model was obtained by replacing each element of the first column of U in [11] with .75 and replacing each of the elements of the second and third columns with .125. In this second allocation matrix only 75 per cent of each resource is to be allocated to the producted of X_1. The sum of the ratios for each row in U must be one since each resource is to be fully allocated to the production of the different products.

When the C vector contained stochastic elements and the dual problem was dealt with by the active approach, then the determination of the ratios in the matrix V depended upon the importance of the constraints in the original primal problem. From the expected value solution of the specific example problem used it was determined that the first constraint is the most significant. The other two constraints each possessed some slack in the optimum solution. The importance of the first constraint in the primal problem was reflected in the dual problem by placing the importance upon the first dual variable W_1. As was the case with the primal problem, the experimental results were dependent upon the specific values assigned to the allocation ratios of the V matrix. Two different sets of values for the allocation ratios of the V matrix were used. In the first run of the experimental model the allocation matrix used for the dual problem is

$$V = \begin{pmatrix} .90 & .05 & .05 \\ .90 & .05 & .05 \\ .90 & .05 & .05 \end{pmatrix} . \qquad [12]$$

The changes made in the allocation matrix V in the second run of the model were similar to the changes made in the allocation matrix U. Again, as was the case with the U matrix, all the allocation ratios in the V matrix must be greater than zero and the sum of the ratios for each row must equal one.

The FORTRAN program of the active approach followed the program of the two-stage approach in the experimental model. Initially the routine embodied in the program determined which parameters in the experimental problem were stochastic. When the stochastic parameters were confined to the C vector the program converted the problem to its dual before the final formulation of the constraints was determined. The program proceeded directly to the final formulation of the constraints in phase one where the stochastic parameters appeared only in the B vector. When both vectors were stochastic the program omitted this section.

In phases one and two of the experiment the number of constraints in the final formulation ranged from five to nine depending upon the number of stochastic parameters in the particular experiment. In the second phase, when the problem was converted to its dual and the sense of the constraints was reversed, both a slack and an artificial variable had to be added to each constraint before using the simplex procedure. Since as many as nine constraints may be involved in any given application of the simplex routine, the number of variables in the simplex tableau can range up to twenty-one. In phase one the maximum number of variables encountered in the simplex procedure was twelve.

Having adapted the particular experimental problem to the requirements of the active approach the program then called the simplex subroutine (SIMPLX) and determined the optimum value of the objective function (ZACT), given that deterministic equivalent. Each experiment was repeated 100 times. The mean (ZACTBR) and the standard deviation (SDZACT) of the optimum objective function values were then determined.

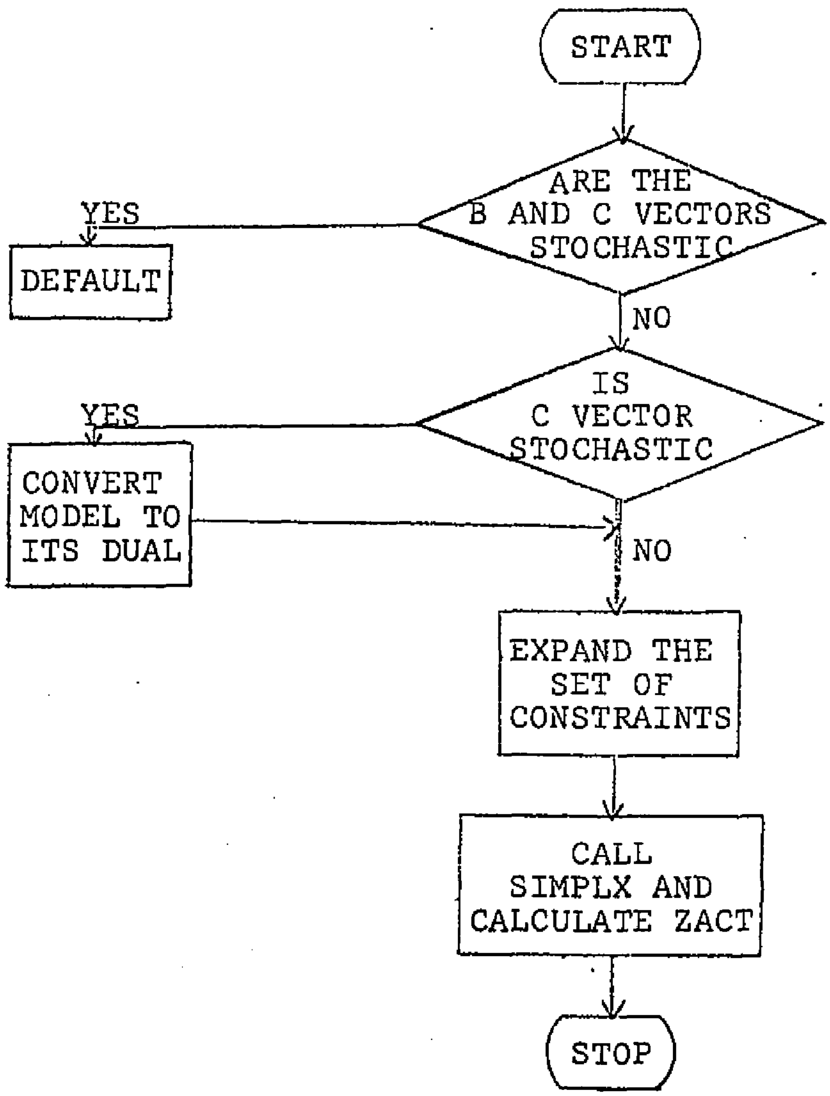
Summary of the Experimental Model

The experimental model developed in this study analyzed 102 different initial formulations of the specific problem used in the experiment. Each formulation was analyzed using a simulation approach (ZSIM), a two-stage approach (ZTWS), an active approach (ZACT), and an expected value approach (ZEXPC). One hundred iterations were performed for each initial formulation with the optimum objective function values recorded for each of the approaches. The differences between the results of the simulation approach and the results of each deterministic equivalent were also determined. The mean and the variance of these differences were then calculated for each deterministic equivalent. These respective means and variances were then used to evaluate the relative efficiency of each of the different deterministic equivalents.

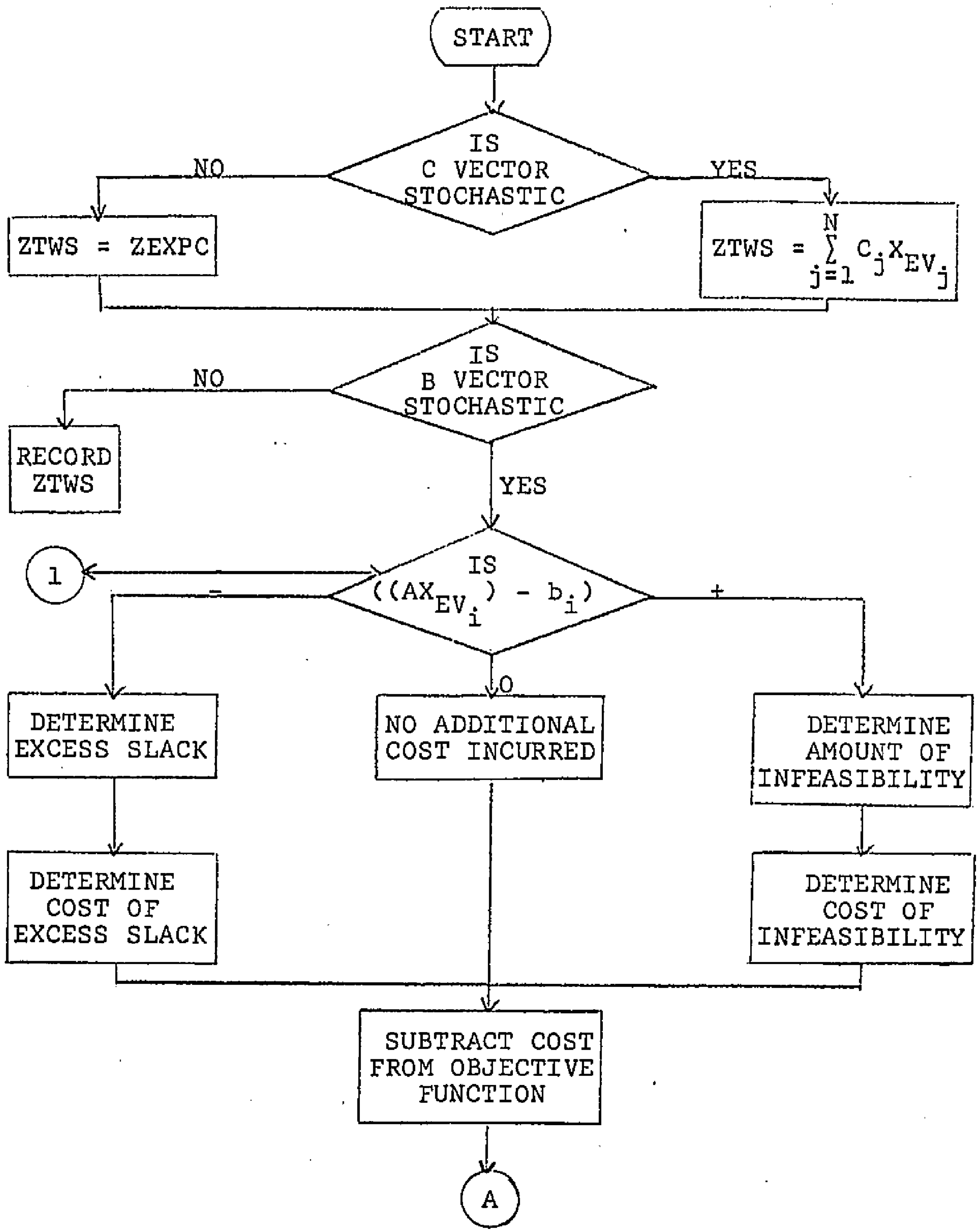
APPENDIX

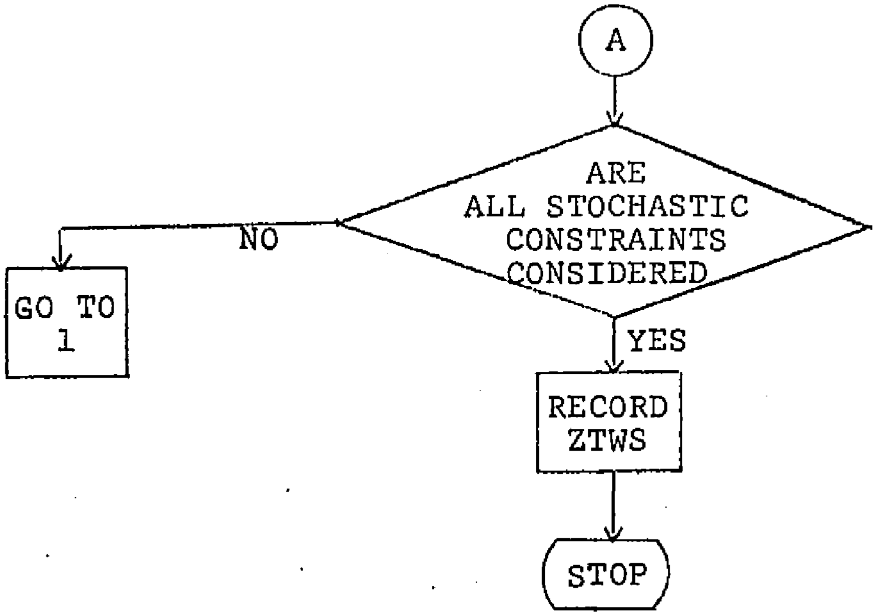
FLOW CHART OF

THE ACTIVE APPROACH

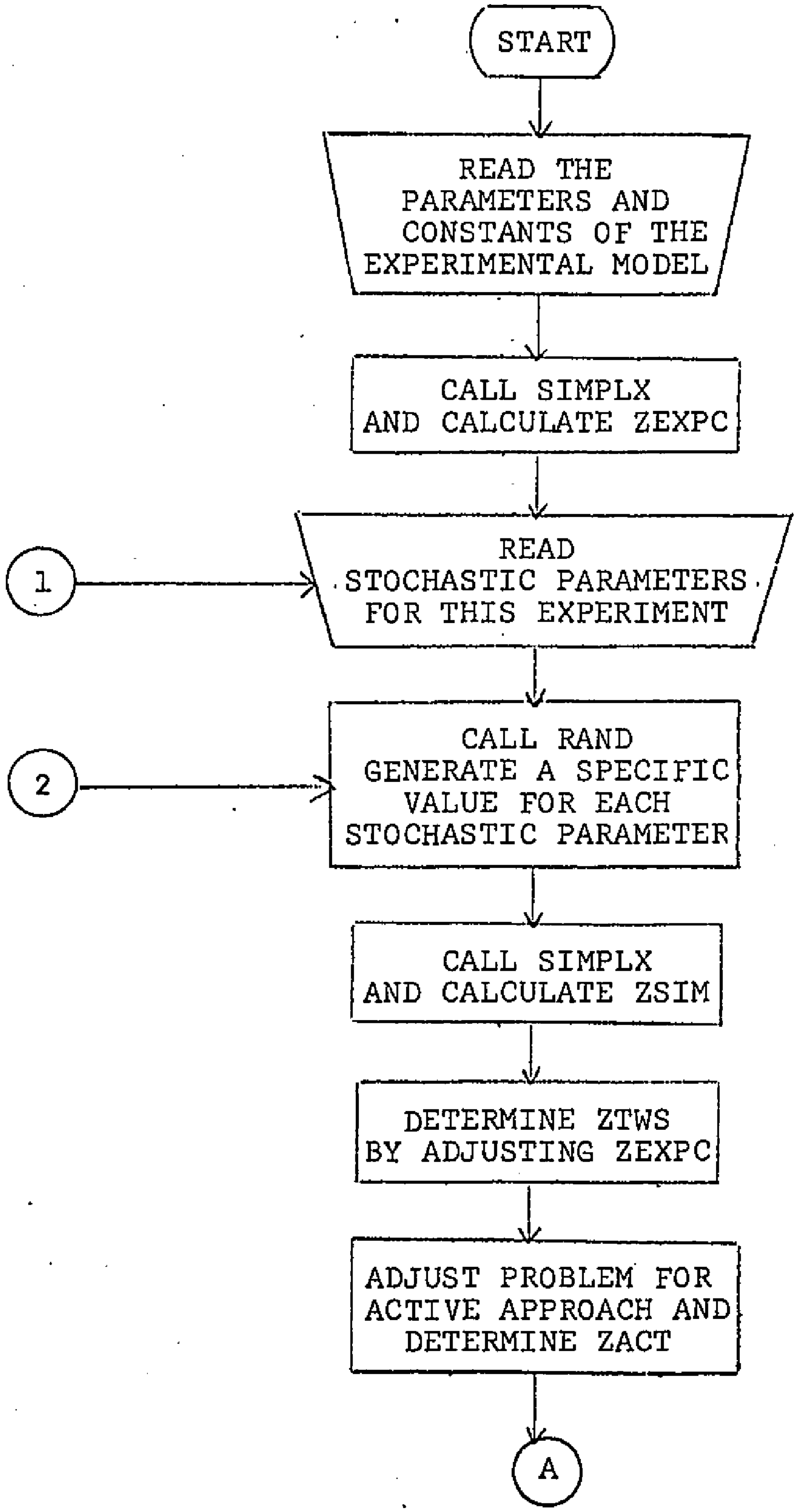


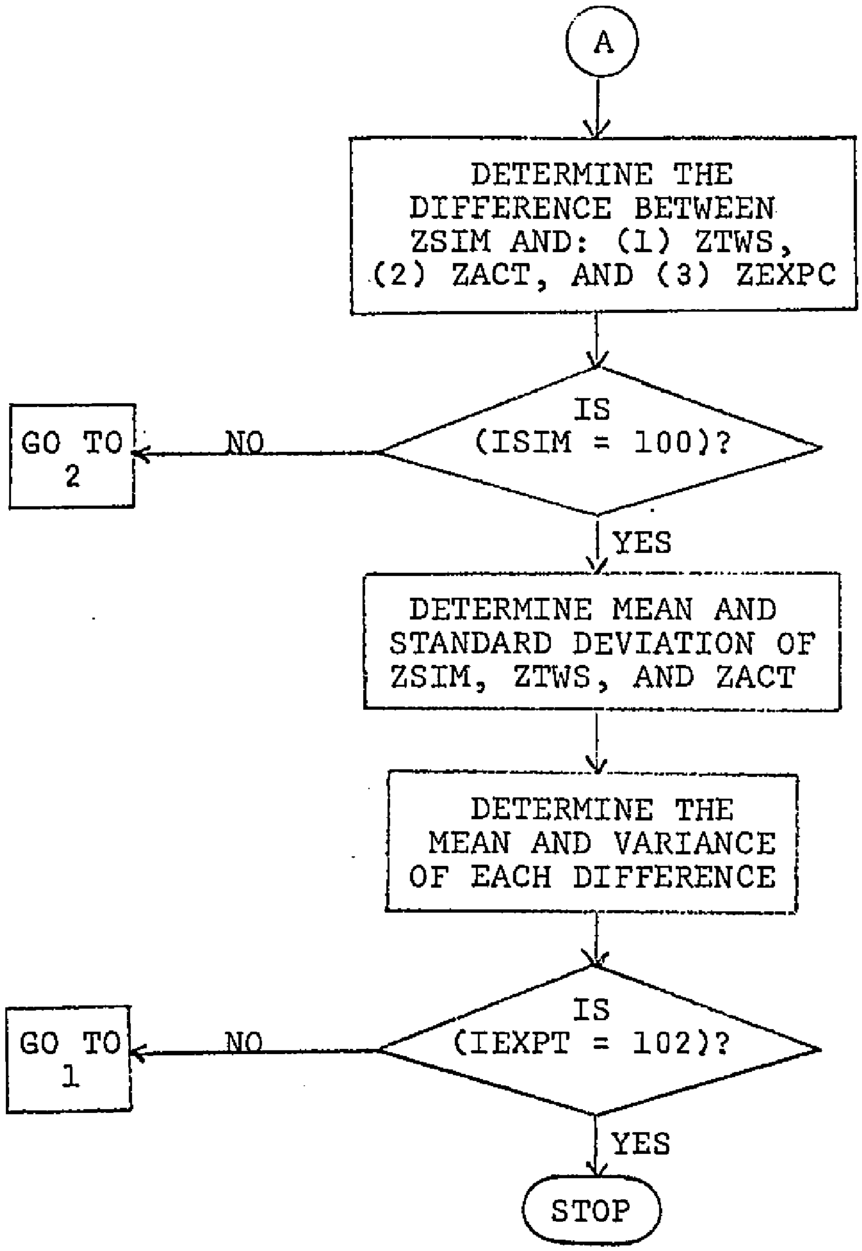
FLOW CHART OF THE TWO-STAGE APPROACH




A
ARE
ALL STOCHASTIC
CONSTRAINTS
CONSIDERED
NO
GO TO
1
YES
RECORD
ZTWS
STOP

FLOW CHART OF THE EXPERIMENTAL MODEL




A
DETERMINE THE DIFFERENCE BETWEEN ZSIM AND: (1) ZTWS, (2) ZACT, AND (3) ZEXPC
IS (ISIM = 100)?
NO
GO TO 2
YES
DETERMINE MEAN AND STANDARD DEVIATION OF ZSIM, ZTWS, AND ZACT
DETERMINE THE MEAN AND VARIANCE OF EACH DIFFERENCE
IS (IEXPT = 102)?
NO
GO TO 1
YES
STOP

```
C           FORTRAN PROGRAM OF THE EXPERIMENTAL MODEL
      DOUBLE PRECISION VAOBFU,A,B,C,AIJ,CJ,DABS,X,BBI,BI
      DOUBLE PRECISION XBAR,PCT,XEV,SAX,AX,STDD,ZSTD,CCJ
      DOUBLE PRECISION ZSIM,ZTWS,ZEXPC,PROF,SPROF,COSINF
      DOUBLE PRECISION COSSLK,SUMZ,SQZ,KKOUNT,AL,BL,CL,U,V
      DIMENSION XEV(6),XBAR(6),STDD(6),AX(3),YPLUS(3),Y(4)
      DIMENSION X(4),SQZ(3),SUMZ(3),Z(4),CSL(3),CIF(3)
      DIMENSION ZSIM(100),ZTWS(100),IX(11),BBI(21),CCJ(21)
      DIMENSION SENSE(21),A(21,21),B(21),C(21),BASIS(21)
      DIMENSION AIJ(21,21),BI(21),CJ(21),BARX(6),STD(6)
      DIMENSION ZIM(100),ZWS(100),ZACT(100),ZCT(100)
      DIMENSION IBVA(3),U(3,3),V(3,3)
      DIMENSION D(100,3),DSQ(100,3),SUDX(3),SUDSQX(3)
      DIMENSION DVAR(102,3),DBAR(102,3)
      COMMON A,AIJ,B,BI,C,CJ,VAOBFU,ZSTD,BASIS,SENSE,IX,MIN,
     1M,N,NN,IERSWT
C     INITILIZING VARIABLES
C     ***************************************************
      DO100I=1,21
      SENSE(I)=0.0
      B(I)=0.0
      BI(I)=0.0
      BBI(I)=0.0
      C(I)=0.0
      CJ(I)=0.0
  100 CONTINUE
      DO101I=1,21
      DO101J=1,21
      A(I,J)=0.0
      AIJ(I,J)=0.0
 101  CONTINUE
      IERSWT=1
C     READING IN DATA PERTAINING TO THE PROGRAMMING MODEL
C     ***************************************************
      KCOUNT=100
      DO2 I=1,11
      IX(I)=0
      READ(5,90) IX(I)
   90 FORMAT(I10)
    2 CONTINUE
      READ(5,1010) MIN
 1010 FORMAT (I1)
      READ(5,1000)M,N
 1000 FORMAT(2I3)
      DO102I=1,M
      READ(5,1001)SENSE(I)
 1001 FORMAT(F2.0)
 102  CONTINUE
 103  READ(5,1002)I,J,XX
 1002 FORMAT(2I5,D15.7)
      IF(I.EQ.99999)GOTO104
      A(I,J)=XX
```

```
      AIJ(I,J)=A(I,J)
      GOTO103
 104  READ(5,1003)I,XX
 1003 FORMAT(I5,D15.7)
      IF(I.GT.M)GOTO105
      B(I)=XX
      BI(I)=B(I)
      BBI(I)=B(I)
      GOTO104
 105  READ(5,1003)J,XX
      IF(J.GT.N)GOTO199
      C(J)=XX
      CJ(J)=C(J)
      GOTO105
  199 READ(5,1012)I,J,XX,YY
 1012 FORMAT(2I5,2D10.9)
      IF(I.EQ.99999) GO TO 200
      U(I,J)=XX
      V(I,J)=YY
      GO TO 199
  200 READ(5,1020)I,XX,YY
 1020 FORMAT(I5,2F10.4)
      IF(I.EQ.99999) GO TO 201
      CSL(I)=XX
      CIF(I)=YY
      GO TO 200
C     CALLING SIMPLEX ROUTINE FOR EXPECTED VALUE SOLUTION
C     ****************************************************************
  201 CALL SIMPLX
      DO1199I=1,NN
      XEV(I)=0.0
 1199 CONTINUE
      DO1198 K=1,NN
      AK=K
      DO1198I=1,M
      IF(BASIS(I).NE.AK)GO TO 1198
      XEV(K)=BI(I)
 1198 CONTINUE
      ZEXPC=VAOBFU
      DO1400IEXPI=1,102
      BSWIT=0.
      CSWIT=0.
      MIN=0
      M=3
      N=3
      DO1013I=1,3
      IBVA(I)=0
 1013 CONTINUE
      DO1409 I=1,KCOUNT
      ZSIM(I)=0.
      ZTWS(I)=0.
      ZACT(I)=0
 1409 CONTINUE
      DO1118I=1,6
```

```
      XBAR(I)=0.
      STDD(I)=0.
      BBI(I)=B(I)
      CCJ(I)=C(I)
 1118 CONTINUE
C     READING DATA PERTAINING TO THE VARICUS EXPERIMENTS
C     **************************************************************
      READ(5,1100)KX
 1100 FORMAT(I1)
      DO1200II=1,KX
      READ(5,1101)XBAR(II),PCT
 1101 FORMAT(D15.7,D5.3)
      STDD(II)=PCT*XBAR(II)
 1200 CONTINUE
C     STARTING THE ITERATIONS FOR THE SIMULATION MODEL
C     **************************************************************
      DO1300ISIM=1,KCOUNT
      DO1215I=1,21
      DO1215J=1,21
      AIJ(I,J)=A(I,J)
 1215 CONTINUE
      DO1216 I=1,21
      BI(I)=B(I)
      CJ(I)=C(I)
 1216 CONTINUE
      BSWIT=0.
      CSWIT=0.
      MIN=0
      M=3
      N=3
      DO1218I=1,21
      SENSE(I)=1.
 1218 CONTINUE
C     DETERMINING THE STOCHASTIC ELEMENTS FOR EACH
C     EXPERIMENT ***************************************************
      DO1220 II=1,KX
      DO1201I=1,3
      IF((DABS(XBAR(II)-B(I))).LT.1.) GO TO 1202
      IF((DABS(XBAR(II)-C(I))).LT. .01) GO TO 1203
 1201 CONTINUE
      IERSWT=7
      GO TO 1401
 1202 BSWIT=1.
      IBVA(I)=1
      CALL RAND
      BI(I)=B(I)+ZSTD*STDD(II)
      BBI(I)=BI(I)
      GO TO 1220
 1203 CSWIT=1.
      IBVA(I)=1
      CALL RAND
      CJ(I)=C(I)+ZSTD*STDD(II)
      CCJ(I)=CJ(I)
 1220 CONTINUE
```

```
C       CALLING THE SIMPLEX ROUTINE FOR THE SIMULATION
C       SOLUTION--ZSIM ****************************************
        CALL SIMPLX
        IF(IERSWT.GT.1)ZSIM(ISIM)=0.0
        IF(IERSWT.GT.1) GO TO 1401
        ZSIM(ISIM)=VAOBFU
C       DETERMINING THE TWO-STAGE SOLUTION--ZTWS
C       *****************************************************
        IF(CSWIT.EQ.1.) GO TO 1204
        PROF=ZEXPC
        GO TO 1206
  1204  PROF=0.
        DO1205K=1,NN
        SPROF=CJ(K)*XEV(K)
        PROF=PROF+SPROF
  1205  CONTINUE
  1206  IF(BSWIT.EQ.1.) GO TO 1207
        ZTWS(ISIM)=PROF
        GO TO 1500
  1207  DO1210I=1,M
        AX(I)=0.
  1210  CONTINUE
        DO1209I=1,M
        DO1209J=1,N
        SAX=A(I,J)*XEV(J)
        AX(I)=AX(I)+SAX
  1209  CONTINUE
        COSLK=0.
        COINF=0.
        DO1211I=1,M
        IF(AX(I)-BBI(I))1212,1213,1214
  1212  YPLUS(I)=BBI(I)-AX(I)-XEV(I+M)
        IF(YPLUS(I).LT.0.) YPLUS(I)=0.
        COS=CSL(I)*YPLUS(I)
        COSLK=COSLK+COS
        GO TO 1211
  1213  YPLUS(I)=0.
        GO TO 1211
  1214  YPLUS(I)=AX(I)-BBI(I)
        COS=CIF(I)*YPLUS(I)
        COINF=COINF+COS
  1211  CONTINUE
        COSINF=COINF
        COSSLK=COSLK
        ZTWS(ISIM)=PROF-COSSLK-COSINF
C       DETERMINING THE ACTIVE RESULT--ZACT
C       *****************************************************
  1500  DO1520I=1,21
        BI(I)=B(I)
        DO1520J=1,21
        CJ(J)=C(J)
        AIJ(I,J)=A(I,J)
  1520  CONTINUE
        ILOOP=0
```

```
      IF((CSWIT*BSWIT).EQ.1.) GO TO 1411
      IF(KX-2)1501,1502,1503
 1501 IP=0
      GO TO 1503
 1502 IP=3
 1503 IF(BSWIT.EQ.1.) GO TO 1510
      IF(MIN.EQ.0) GO TO 1511
      MIN=0
      GO TO1512
 1511 MIN=1
 1512 DO1514I=1,M
      IF(SENSE(I).EQ.1.) GO TO 1513
      SENSE(I)=1.
      GO TO 1514
 1513 SENSE(I)=2.
 1514 CONTINUE
      DO1516I=1,M
      CJ(I)=B(I)
 1516 CONTINUE
      F=N
      N=M
      M=F
 1510 DO1504I=1,3
      IF(IBVA(I).EQ.1) GO TO 1506
      IP=IP+1
      MM=M+IP
      DO1505J=1,3
      IF(CSWIT.EQ.1.) GO TO 1523
      AIJ(MM,J)=A(I,J)
      GO TO 1505
 1523 AIJ(MM,J)=A(J,I)
 1505 CONTINUE
      IF(CSWIT.EQ.1.) GO TO 1524
      BI(MM)=B(I)
      SENSE(MM)=SENSE(I)
      GO TO 1504
 1524 BI(MM)=C(I)
      SENSE(MM)=SENSE(I)
      GO TO 1504
 1506 DO1507II=1,N
      IF(ILOOP.EQ.0) IQ=0
      IF(ILOOP.EQ.1) IQ=M
      IF(ILOOP.EQ.2) IQ=2*M
      IM=IQ+II
      IF(CSWIT.EQ.1.) GO TO 1521
      BI(IM )=BBI(I)*U(I,II)
      GO TO 1522
 1521 BI(IM )=CCJ(I)*V(I,II)
 1522 SENSE(IM )=SENSE(I)
      DO1507J=1,N
      IF(II.EQ.J) GO TO 1508
      AIJ(IM ,J)=0.
      GO TO 1507
 1508 IF(CSWIT.EQ.1.) GO TO 1525
```

```
      AIJ(IM ,J)=A(I,J)
      GO TO 1507
 1525 AIJ(IM,J)=A(J,I)
 1507 CONTINUE
      ILOOP=ILOOP+1
 1504 CONTINUE
      IF(KX.EQ.3)MM=9
      M=MM
      CALL SIMPLX
      ZACT(ISIM)=VAOBFU
      IF(IERSWT.GT.1)ZACT(ISIM)=0.0
 1411 IF(IERSWT.EQ.1) GO TO 1300
C     EXITS TO BE TAKEN WHEN AN ERROR RESULTS IN THE SIMPLEX
C     ROUTINE**********************************************
 1401 GO TO (1410,1403,1404,1405,1406,1407,1408,1402),IERSWT
 1402 WRITE(6,1004)
 1004 FORMAT(1X,'ERROR IN INITIAL TABLEAU O NO POSITIVE ONE'
     1,' APPEARS IN THIS ROW')
      GO TO 1300
 1403 WRITE(6,1005)
 1005 FORMAT(1X,'ERROR IN INITIAL TABLEAU O THIS COLUMN (SL',
     1'ACK OR ARTIFICIAL) HAS MORE THAN ONE UNIT ELEMENT')
      GO TO 1300
 1404 WRITE(6,1102)
 1102 FORMAT(1X,'ERROR O MORE VARIABLES ARE IN BASIS THAN',
     1'THERE ARE CONSTRAINTS')
      GO TO 1300
 1405 WRITE(6,5020)
 5020 FORMAT(1X,'SOLUTION IS UNBOUNDED O NO AIJ(I,JK) IS',
     1'POSITIVE')
      GO TO 1300
 1406 WRITE(6,5032)
 5032 FORMAT(1X,'PERTURBED CONSTRAINTS ARE STILL TIED...',
     1'CONSTRAINTS ARE LINEARLY DEPENDENT')
      GO TO 1300
 1407 WRITE(6,5040)
 5040 FORMAT(1X,'PROGRAM MAY BE CYCLING VARIABLE HAS ENTERED
     1 10 TIMES')
      GO TO 1300
 1408 WRITE(6,1006)
 1006 FORMAT(/1X,'ERROR  NO PARAMETERS ARE STOCHASTIC')
      GO TO 1300
 1410 WRITE(6,1009)
 1009 FORMAT(/1X,'ERROR. SWITCH SHOULD NOT BRANCH TO 1401
     1 WHEN IERSWT= 1')
 1300 CONTINUE
C     DETERMINING THE MEAN AND THE VARIANCES OF THE
C     DIFFERENCES ******************************************
      DO1903I=1,3
      SUDX(I)=0.0
      SUDSQX(I)=0.0
 1903 CONTINUE
      DO1900I=1,KCOUNT
      D(I,1)=ZSIM(I)-ZTWS(I)
```

```
      D(I,2)=ZSIM(I)-ZACT(I)
      D(I,3)=ZSIM(I)-ZEXPC
      DSQ(I,1)=D(I,1)**2
      DSQ(I,2)=D(I,2)**2
      DSQ(I,3)=D(I,3)**2
 1900 CONTINUE
      DO1901J=1,3
      DO1901I=1,KCOUNT
      SUDX(J)=SUDX(J)+D(I,J)
      SUDSQX(J)=SUDSQX(J)+DSQ(I,J)
 1901 CONTINUE
      DO1902J=1,3
      DBAR(IEXPI,J)=SUDX(J)/100.
      DVAR(IEXPI,J)=(100.*SUDSQX(J)-SUDX(J)**2)/(100.*99.)
 1902 CONTINUE
C     DETERMINING THE MEAN, VARIANCE, AND STANDARD DEVIATION
C     OF ZSIM, ZTWS, AND ZACT *******************************
      DO1301 I=1,3
      SQZ(I)=0.
      SUMZ(I)=0.
 1301 CONTINUE
      DO1302I=1,KCOUNT
      SUMZ(1)=SUMZ(1)+ZSIM(I)
      SUMZ(2)=SUMZ(2)+ZTWS(I)
      SUMZ(3)=SUMZ(3)+ZACT(I)
      SQZ(1)=SQZ(1)+ZSIM(I)**2
      SQZ(2)=SQZ(2)+ZTWS(I)**2
      SQZ(3)=SQZ(3)+ZACT(I)**2
 1302 CONTINUF
      KKOUNT=KCOUNT
      ZSIMBR=SUMZ(1)/KKOUNT
      ZTWSBR=SUMZ(2)/KKOUNT
      ZACTBR=SUMZ(3)/KKOUNT
      VRZSIM=(KKOUNT*SQZ(1)-SUMZ(1)**2)/KKOUNT**2
      VRZTWS=(KKOUNT*SQZ(2)-SUMZ(2)**2)/KKOUNT**2
      VRZACT=(KKOUNT*SQZ(3)-SUMZ(3)**2)/KKOUNT**2
      SDZSIM=SQRT(VRZSIM)
      SDZTWS=SQRT(VRZTWS)
      SDZACT=SQRT(VRZACT)
C     THE CHI-SQUARE TEST OF NORMALITY ON ZSIM
C     **************************************************
      AL=ZSIMBR-.6745*SDZSIM
      BL=ZSIMBR
      CL=ZSIMBR+.6745*SDZSIM
      DO1309I=1,4
      Y(I)=0.
 1309 CONTINUE
      DO1307I=1,KCOUNT
      IF(ZSIM(I).LT.AL) GO TO 1303
      IF(ZSIM(I).LT.BL) GO TO 1304
      IF(ZSIM(I).LT.CL) GO TO 1305
      J=4
      GO TO 1306
 1303 J=1
```

```
      GO TO 1306
 1304 J=2
      GO TO 1306
 1305 J=3
 1306 Y(J)=Y(J)+1.
 1307 CONTINUE
      CHISUM=0.
      DO1308I=1,4
      CHISUM=CHISUM+(Y(I)-25.)**2
 1308 CONTINUE
      CHISIM=.04*CHISUM
C     THE CHI-SQUARE TEST OF NORMALITY ON ZTWS
C     ****************************************************************
      AL=ZTWSBR-.6745*SDZTWS
      BL=ZTWSBR
      CL=ZTWSBR+.6745*SDZTWS
      DO1310I=1,4
      X(I)=0.
 1310 CONTINUE
      DO1311I=1,KCOUNT
      IF(ZTWS(I).LT.AL) GO TO 1312
      IF(ZTWS(I).LT.BL) GO TO 1313
      IF(ZTWS(I).LT.CL) GO TO 1314
      J=4
      GO TO 1315
 1312 J=1
      GO TO 1315
 1313 J=2
      GO TO 1315
 1314 J=3
 1315 X(J)=X(J)+1.
 1311 CONTINUE
      CHISUM=0.
      DO1316I=1,4
      CHISUM=CHISUM+(X(I)-25.)**2
 1316 CONTINUE
      CHITWS=.04*CHISUM
C     CHI-SQUARE TEST OF NORMALITY ON ZACT
C     ****************************************************************
      AL=ZACTBR-.6745*SDZACT
      BL=ZACTBR
      CL=ZACTBR+.6745*SDZACT
      DO1317I=1,4
      Z(I)=0.
 1317 CONTINUE
      DO1318I=1,KCOUNT
      IF(ZACT(I).LT.AL) GO TO 1319
      IF(ZACT(I).LT.BL) GO TO 1320
      IF(ZACT(I).LT.CL) GO TO 1321
      J=4
      GO TO 1322
 1319 J=1
      GO TO 1322
 1320 J=2
```

```
      GO TO 1322
 1321 J=3
 1322 Z(J)=Z(J)+1
 1318 CONTINUE
      CHISUM=0.
      DO1323I=1,4
      CHISUM=CHISUM+(Z(I)-25.)**2
 1323 CONTINUE
      CHIACT=.04*CHISUM
C     PRINT OUT STSTEMENTS
C     ****************************************************
      WRITE(6,1007)IEXPI,KX
 1007 FORMAT(/19X,'EXPERIMENT NUMBER',3X,I3,17X,'NUMBER OF',
     1' STOCHASTIC PARAMETERS',3X,I2)
      WRITE(6,1103)
 1103 FORMAT(///13X,'PROPERTIES OF STOCHASTIC PARAMETERS')
      WRITE(6,1104)
 1104 FORMAT(/25X,'PARAMETER',5X,'MEAN',7X,'STANDARD',
     1' DEVIATION')
      DO1008I=1,KX
      BARX(I)=XBAR(I)
      STD(I)=STDD(I)
 1008 CONTINUE
      WRITE(6,1105)(BARX(I),STD (I),I=1,KX)
 1105 FORMAT(/37X,F12.4,7X,F12.4)
      WRITE(6,1110)
 1110 FORMAT(///23X,'PROPERTIES OF THE ALTERNATIVE',
     1' DETERMINISTIC EQUIVALENTS')
      WRITE(6,1111)
 1111 FORMAT(/30X,'SIMULATION',6X,'TWO-STAGE',10X,'ACTIVE',
     16X,'EXPECTED VALUE')
      WRITE(6,1112)
 1112 FORMAT(31X,'APPROACH',8X,'APPROACH',9X,'APPROACH',9X,
     1'APPROACH')
      ZEXC=ZEXPC
      WRITE(6,1113)ZSIMBR,ZTWSBR,ZACTBR,ZEXC
 1113 FORMAT(/17X,'MEAN',9X,F10.4,6X,F10.4,6X,F10.4,6X,
     1F10.4)
      WRITE(6,1114)VRZSIM,VRZTWS,VRZACT
 1114 FORMAT(/17X,'VARIANCE',4X,F11.3,5X,F11.3,5X,F11.3,10X,
     1'NA')
      WRITE(6,1115)SDZSIM,SDZTWS,SDZACT
 1115 FORMAT(/17X,'STD. DEV',5X,F10.3,6X,F10.3,6X,F10.3,10X,
     1'NA')
      WRITE(6,1116) CHISIM,CHITWS,CHIACT
 1116 FORMAT(/17X,'CHI SQ TEST',2X,F10.5,6X,F10.5,6X,F10.5,
     110X,'NA')
      WRITE(6,1117)
 1117 FORMAT(/17X,'ALPHA =.05      7.81473 AL=.01  11.3449')
      WRITE(6,1106)
 1106 FORMAT('*********************************************'
     1'**************************************************')
      WRITE(6,1920)DBAR(IEXPI,1),DBAR(IEXPI,2),DBAR(IEXPI,3)
 1920 FORMAT(22X,F11.2,11X,F11.2,11X,F11.2)
```

```
      WRITE(7,1920)DBAR(IEXPI,1),DBAR(IEXPI,2),DBAR(IEXPI,3)
      WRITE(6,1920)DVAR(IEXPI,1),DVAR(IEXPI,2),DVAR(IEXPI,3)
      WRITE(7,1920)DVAR(IEXPI,1),DVAR(IEXPI,2),DVAR(IEXPI,3)
 1400 CONTINUE
      STOP
      END
C
C     THE RANDOM NUMBER GENERATOR SUBROUTINE PRODUCING A
C     STANDARD NORMAL DEVIATE *****************************
      SUBROUTINE RAND
      DOUBLE PRECISION ZSTD,A,AIJ,B,BI,C,CJ,VAOBFU
      DIMENSION IX(11),R(2),AIJ(21,21),BI(21),CJ(21)
      DIMENSION SENSE(21),A(21,21),B(21),C(21),BASIS(21)
      COMMON A,AIJ,B,BI,C,CJ,VAOBFU,ZSTD,BASIS,SENSE,IX,MIN,
     1M,N,NN,IERSWT
      DO22 II=1,2
      I=11
   20 IIX=IX(I)
      IY=IIX* 46331
      IF(IY)5,6,6
    5 IY=IY+2147483647 +1
    6 YFL=IY
      IX(I)=IY
      YFL=YFL*.4656613E-9
      IF(I.NE.11)GO TO 21
      IF(YFL.LT..1) GO TO 11
      IF(YFL.LT..2) GO TO 12
      IF(YFL.LT..3) GO TO 13
      IF(YFL.LT..4) GO TO 14
      IF(YFL.LT..5) GO TO 15
      IF(YFL.LT..6) GO TO 16
      IF(YFL.LT..7) GO TO 17
      IF(YFL.LT..8) GO TO 18
      IF(YFL.LT..9) GO TO 19
      I=10
      GO TO 20
   11 I=1
      GO TO 20
   12 I=2
      GO TO 20
   13 I=3
      GO TO 20
   14 I=4
      GO TO 20
   15 I=5
      GO TO 20
   16 I=6
      GO TO 20
   17 I=7
      GO TO 20
   18 I=8
      GO TO 20
   19 I=9
      GO TO 20
```

```
   21 R(II)=YFL
   22 CONTINUE
      ZSTD=SQRT(-2.*ALOG(R(1)))*COS(2.*3.142857*R(2))
      RETURN
      END
C
C     THE SIMPLEX SUBROUTINE FOR SOLVING A LINEAR PROGRAMMING
C     MODEL ***********************************************
      SUBROUTINE SIMPLX
      DOUBLE PRECISION  VAOBFU,ENTR,BIPYM,PAIJ,SUPAIJ,PBI
      DOUBLE PRECISION A,AIJ,B,BI,C,CJ,BASCJ,ZJCJ,ZJ,SUMZJ
      DOUBLE PRECISION AIJPI,BIPI,SUBENT,ALARGE,SMALL,DABS
      DOUBLE PRECISION ZSTD,PIVOT,VALU,SUVALU,X
      DIMENSION SENSE(21),A(21,21),B(21),C(21),BASIS(21)
      DIMENSION ZJCJ(21),ZJ(21),SUMZJ(21),ENTR(21),BIPYM(21)
      DIMENSION PAIJ(21),SUPAIJ(21),PBI(21),CYCLE(21)
      DIMENSION AIJPI(21,21),BIPI(21),AIJ(21,21),BI(21)
      DIMENSION BASCJ(21),CJ(21),IX(11)
      COMMON A,AIJ,B,BI,C,CJ,VAOBFU,ZSTD,BASIS,SENSE,IX,MIN,
     1M,N,NN,IERSWT
C     INITIALIZING VARIABLES*********************************
      DO100I=1,21
      BASIS(I)=0.0
      BASCJ(I)=0.0
      ZJCJ(I)=0.0
      ZJ(I)=0.0
      SUMZJ(I)=0.0
      BIPYM(I)=0.0
      PAIJ(I)=0.0
      SUPAIJ(I)=0.0
      PBI(I)=0.0
      CYCLE(I)=0.0
  100 CONTINUE
      IERSWT=1
      NS=0
      NA=0
C     EXPANDING INITIAL TABLEAU ****************************
  199 DO210I=1,M
      K=1
      IF(SENSE(I)-2.)200,201,202
  200 NS=NS+1
      AIJ(I,N+NS+NA)=1.0
      CJ(N+NS+NA)=0.0
      GOTO209
  201 NS=NS+1
      AIJ(I,N+NS+NA)=-1.0
      CJ(N+NS+NA)=0.0
      NA=NA+1
      IF(MIN.EQ.1) GO TO 720
      AIJ(I,N+NS+NA)=1.0
      CJ(N+NA+NS)=-9999.
      GOTO209
  720 AIJ(I,N+NS+NA)=1.0
      CJ(N+NS+NA)=9999.
```

```
      GO TO 209
  202 NA=NA+1
      IF(MIN.EQ.1) GO TO 721
      AIJ(I,N+NS+NA)=1.0
      CJ(N+NA+NS)=-9999.
      GO TO 209
  721 AIJ(I,N+NS+NA)=1.0
      CJ(N+NS+NA)=9999.
 209  IF(I.GE.M)GOTO300
 210  CONTINUE
  300 NN=N+NS+NA
C     FINDING THE INITIAL BASIS ******************************
      JJ=1
      DO320L=1,M
      I=L
      J=N+1
 301  IF(AIJ(I,J).EQ.1.)GOTO302
      J=J+1
      IF(J.LE.NN) GO TO 301
      IERSWT=8
      GO TO 998
 302  IF(I.EQ.1)GOTO303
      KK=I
      I=0
 311  I=I+1
      IF(I.NE.KK)GOTO304
      GOTO305
 303  I=I+1
 304  IF(A(I,J).EQ.0.0)GOTO305
      IERSWT=2
      GO TO 998
 305  IF(I.GE.M)GOTO308
      IF(L.EQ.1)GOTO303
      GOTO311
  308 AJ=J
      IF(JJ.GT.M)GOTO312
      BASIS(JJ)=AJ
      JJ=JJ+1
      GOTO320
  312 IERSWT=3
      GO TO 998
 320  CONTINUE
C     FINDING COST VECTOR FOR BASIS VARIABLES **************
 400  DO401JJ=1,M
      JJK=BASIS(JJ)
      BASCJ(JJ)=CJ(JJK)
 401  CONTINUE
C     FINDING ZJCJ(J).INDICATOR ROW AND VALUE OF OBJECTIVE
C     FUNCTION ******************************************
  603 DO 604 J=1,NN
      ENTR(J)=0.0
      ZJ(J) =0.0
      SUMZJ(J) =0.0
      ZJCJ(J) =0.0
```

```
 604  CONTINUE
      DO740I=1,M
      BIPYM(I)=999999.
      PAIJ(I)=0.0
      SUPAIJ(I)=C.0
      PBI(I)=0.0
      BIPI(I)=0.0
  740 CONTINUE
      KMARK=0
      DO741I=1,M
      DO741J=1,NN
      AIJPI(I,J)=0.0
  741 CONTINUE
      DO402J=1,NN
      DO402I=1,M
      ZJ(J)=AIJ(I,J)*BASCJ(I)
      SUMZJ(J)=SUMZJ(J)+ZJ(J)
 402  CONTINUE
      DO403J=1,NN
      ZJCJ(J)=SUMZJ(J)-CJ(J)
 403  CONTINUE
      VAOBFU = 0.0
      DO420I=1,M
      VALU=BI(I)*BASCJ(I)
      VAOBFU = VAOBFU + VALU
 420  CONTINUE
C     TESTING FOR OPTIMAL SOLUTION ************************
      IF(MIN .EQ.1) GO TO 730
      DO404J=1,NN
      IF(ZJCJ(J).LT.0.0)ENTR(J)=ZJCJ(J)
 404  CONTINUE
      GO TO 732
  730 DO 731 J=1,NN
      IF (ZJCJ(J).GT.C.0) ENTR(J)=ZJCJ(J)
  731 CONTINUE
  732 DO 405 J=1,NN
      IF(ENTR(J).NE.0.0)GOTO406
 405  CONTINUE
      IERSWT=1
  998 RETURN
  406 ALARGE = 0.0
C     TESTING FOR ENTERING VARIABLE ************************
      DO407J=1,NN
      SUBENT = ENTR(J)
      IF(DABS(ALARGE).GE.DABS(SUBENT))  GO TO 407
      ALARGE =DABS(SUBENT)
      JK =J
 407  CONTINUE
C     FINDING VARIABLE TO LEAVE SOLUTION *******************
      MARKER=0
      DO503I=1,M
      IF(AIJ(I,JK).GT.0.0)GOTO501
      BIPYM(I)=999999.
      GOTO503
```

```
 501   MARKER=1
       BIPYM(I)=BI(I)/AIJ(I,JK)
 503   CONTINUF
       IF(MARKER.EQ.1)GOTO504
       IFRSWT=4
       GO TO 998
  504  SMALL=BIPYM(1)
       IR=1
       DO506I=2,M
       IF(SMALL.LE.BIPYM(I))GOTO506
       SMALL=BIPYM(I)
       IR=I
 506   CONTINUE
       DO505I=1,M
       IF(I.EQ.IR) GO TO 5C5
       IF((DABS(SMALL-BIPYM(I))).LT.  .COOOOC9) KMARK=1
 505   CONTINUE
       IF(KMARK.NE.1)GOTO510
       IRK =IR
       DO509I=1,M
       IF(I.EQ.IRK) GO TO 509
       IF((DABS(SMALL-BIPYM(I))).GT.  .COOOCO9) GO TO 509
       WRITE(6,5030)
 5030  FORMAT(1X,'SOLUTICN IS DEGFNERATE')
C      PERTURBING THE TIED BIPYM(I) VECTOR *****************
       E=.01
       DO 750 K=1,M
       PAIJ(K)=0.0
       SUPAIJ(K)=0.0
       PBI(K)=0.0
  750  CONTINUE
       DO508J=1,NN
       PAIJ(IR)=E**J*AIJ(IR,J)
       SUPAIJ(IR)=SUPAIJ(IR)+PAIJ(IR)
       PAIJ(I)=E**J*AIJ(I,J)
       SUPAIJ(I)=SUPAIJ(I)+PAIJ(I)
 508   CONTINUE
       PBI(IR)=BI(IR)+SUPAIJ(IR)
       PBI(I)=BI(I)+SUPAIJ(I)
       BIPYM(I) = PBI(I)/AIJ(I,JK)
       BIPYM(IR) = PBI(IR)/AIJ(IR,JK)
       IF((DABS(BIPYM(I)-BIPYM(IR))).LT.1.0D-17) GO TO 511
       IF(BIPYM(I).LT.BIPYM(IR)) GO TO 760
       KKK=I
       GO TO 509
  760  KKK=IR
       IR=I
 509   CONTINUE
       GO TO 510
  511  IERSWT=5
       GO TO 998
  510  BASIS(IR)=JK
       BASCJ(IR)=CJ(JK)
       CYCLE(JK)=CYCLE(JK)+1.0
```

```
      IF(CYCLE(JK).LE.10.)GOTO600
      IERSWT=6
      GO TO 998
C     TRANSFORMING THE SIMPLEX TABLEAU ********************
  600 PIVOT = AIJ(IR,JK)
      DO601J=1,NN
      AIJPI(IR,J)=AIJ(IR,J)/PIVOT
 601  CONTINUE
      BIPI(IR)=BI(IR)/PIVOT
      DO 700 I=1,M
      IF(I.EQ.IR)GOTO700
      BIPI(I) =BI(I)-(AIJ(I,JK)*BIPI(IR))
  700 CONTINUE
      IF(KMARK.EQ.1) BIPI(KKK)=0.0
      DO602 I=1,M
      IF(I.EQ.IR) GO TO 602
      DO 701 J=1,NN
      AIJPI(I,J)=AIJ(I,J)-(AIJ(I,JK)*AIJPI(IR,J))
  701 CONTINUE
 602  CONTINUE
      DO 705 J=1,NN
      DO 705 I=1,M
      AIJ(I,J)=AIJPI(I,J)
  705 CONTINUE
      DO 706 I=1,M
      BI(I)=BIPI(I)
  706 CONTINUE
      GOTO603
      END
```

CHAPTER V

ANALYSIS OF THE EXPERIMENTAL RESULTS

Introduction

The objective of this study, as it is stated in the first chapter, has been to determine the efficiency of using various alternative deterministic equivalents in order to solve a stochastic programming model. To achieve this objective a simulation of the stochastic model was used as a standard for comparison. For each set of initial conditions the expectations of the optimum objective function values which have been determined by utilizing each deterministic equivalent were compared to the optimum objective function values determined from a simulation of the model.

This chapter presents an analysis of the results generated by the experimental model used in this study. These results are separated into two parts, reflecting the two sets of results obtained for the two problems used.

The next section of this chapter includes a statement of the statistical test which has been performed on the results of the study. In this section special emphasis is placed upon the organization of the results of the experiments, the statement of the hypotheses which are tested, and the assumptions upon which the statistical tests are based. The following section presents analyses of the results of the statistical tests, given the various assumptions made concerning the initial formulations of the model. Each deterministic equivalent of the stochastic programming model was tested in each phase of the experiment as the coefficients of variation of the stochastic parameters changed. The final section of this chapter states the conclusions which have been drawn from this study and indicates some aspects of the problem which require further experimentation.

In the appendix to the chapter the experimental results are presented in tabular form. Tables 2 through 4 present, for problem A, the sample means of the distributions of the optimum objective function values for each deterministic equivalent as well as for the simulation approach. Tables 5 through 7, present a similar set of results for problem B. In Table 8 and Table 9 the results from the application of the statistical tests are presented for problems A and B respectively. Tables 10 through 12 summarize, for each deterministic equivalent, and for each phase of the experiment, the findings relevant to the cases when (1) the solutions generated by each of the deterministic equivalents were on the average feasible and (2) when the results led to an acceptance of the hypothesis of no difference between the simulation approach and the particular deterministic equivalent.

The Statistical Tests

The characteristics of the experimental results

In each phase of the experiment each initial formulation of the empirical problem has been used to generate a series of optimum objective function values for the simulation approach and for each deterministic equivalent. For each of the 102 initial formulations of the problem 100 iterations were performed. Thus 100 values of ZSIM, ZTWS(20), ZTWS(40), ZTWS(80), ZACT(.90), and ZACT(.75) were generated.[1] For each of the 102 experiments the mean values ZSIMBR, ZTWSBR(20), ZTWSBR(40), ZTWSBR(80), ZACTBR(.90), and ZACTBR(.75) which were determined are presented in Tables 2 through 7 in the appendix. The optimum value of the objective function, ZEXPC, determined from the expected value approach is also included in the tables mentioned above.

For any given initial formulation of the problem, the sample of the optimum objective function values

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[1]The term ZTWS(20) refers to the two-stage approach utilizing the set of cost vectors, which includes 20 as the first cost coefficient in the vector COSLK. The terms ZTWS(40) and ZTWS(80) respectively refer to the two-stage approach utilizing the second and third sets of cost vectors COSLK and COINF. Similarly, ZACT(.90) and ZACT(.75) refer to the active approach utilizing the different sets of allocation ratios indicated.

generated for each deterministic equivalent is not necessarily independent of the corresponding sample generated by the simulation approach for this initial formulation. This statement is supported by the fact that on each iteration of the experiment, once the values for the stochastic parameters were determined, then those values were used to generate an optimum solution for each deterministic equivalent and for the simulation approach. In effect, then, dependent samples were used in the statistical tests performed in this study.

A statement of the hypothesis and the statistical test[2]

The statistical test performed in this study is designed to test the hypothesis of the equality of two population means under the conditions that the populations from which the samples were taken are normally distributed and that the individual samples are not independent.

For each deterministic equivalent tested in the experiments, d_{ij} is defined as

$$d_{ij} = ZSIM_j - x_{ij} \quad . \qquad [1]$$

The value, x_{ij}, is the optimum objective function value determined on the jth iteration by using the ith

---

[2]John E. Freund, Paul E. Livermore, and Irwin Miller, Manual of Experimental Statistics (Englewood Cliffs, N. J.; Prentice-Hall, Inc., 1960), pp. 19-21; and I. M. Chakravarti, R. G. Laha, and J. Roy, Handbook of Methods of Applied Statistics (New York: John Wiley and Sons, Inc., 1967), pp. 325-326.

deterministic equivalent, while $ZSIM_j$ is the optimum objective function value determined on this iteration by using the simulation approach. Since six variations of the different deterministic equivalents were considered the i subscript ranged from 1 to 6. The j subscript which referred to the number of iterations ranged from 1 to 100. A value of d_{ij} was determined for all deterministic equivalents on each iteration of the experiments.

For each experiment the mean and the variance of the d_{ij}'s were defined as

$$\bar{d}_{ih} = \frac{\sum_{j=1}^{100} d_{ij}}{100} \quad , \quad \text{and}$$

$$s^2_{d_{ih}} = \frac{100 \sum_{j=1}^{100} d^2_{ij} - (\sum d_{ij})^2}{100 \cdot 99} \quad . \qquad [2]$$

In the equations in [2] i refers to the different deterministic equivalents, h refers to the different experiments which were performed, and j refers to the number of iterations of each experiment.

The statistical tests were performed on the $\bar{d}_{ih}$ values, which are the means of the differences between the paired sample results. These mean values were determined by averaging the differences between the optimum objective function values generated by the simulation approach and by each deterministic equivalent.

These optimum objective function values are assumed to be dependent since on any iteration the same values of the stochastic parameters were used to generate a solution for each approach.

The distributions of these different mean values are normal. The reader should recall that through an application of the Central Limit Theorem it can be assumed that, as the size of the sample increases, the distribution of the means of a set of samples randomly and independently drawn from a given population will approach a normal distribution regardless of the shape of the parent population.

The general model of the statistical test can be stated as

$$d_j = x_{1j} - x_{2j} = \mu_1 - \mu_2 + \varepsilon_j \qquad [3]$$

where ε_j is a normally distributed error term with a mean equal to zero and a standard deviation equal to the standard deviation of the population of the d_j's.

The Z-statistic[3] can be defined as

$$Z_i = \frac{\bar{\bar{d}}_i \sqrt{n}}{s_{d_t}} \qquad [4]$$

---

[3]When the expected value approach is tested then x_{ij} in equation [1] is constant for all values of j. In this case the statistic Z in equation [4] is equivalent to

$$Z = \frac{(\bar{\bar{x}} - a)\sqrt{n}}{s_{d_i}}$$

where a is the constant value, ZEXPC, $\bar{\bar{x}}$ is the mean of the where $$\bar{\bar{d}}_i = \frac{\sum_{h=1}^{k} \bar{d}_{ih}}{k} ,$$

$$s_{d_t}^2 = \sum_{h=1}^{k} s_{d_{ih}}^2 \quad \text{, and where} \qquad [5]$$

$\bar{d}_{ih}$ and $s_{d_{ih}}^2$ are given in the equations in [2]. The second equation in [5] indicates that $s_{d_t}^2$ is a pooled variance which is determined from the sample variances of the d_{ij}'s. Although the population variance is unknown, a Z test is appropriate due to the large sample sizes resulting from each experiment.

In the first two phases of the experimental procedure four initial formulations of the experimental problems were evaluated as the coefficients of variation of the stochastic parameters changed. In the third phase nine initial formulations were tested. In the first two phases, the value of h in equation [5] ranged from one to four and n = 400; while in the third phase, the value of h ranged from one to nine and n = 900. For each experimental problem considered, six statistical tests were performed in each phase on the different variations of the deterministic equivalents which were evaluated.

---

h values of ZSIMBR and s_{d_i} is the sample standard deviation of all n ZSIM values included. In this case the null hypothesis states that the mean of the population from which the samples of ZSIMBR values are taken is equal to the constant value ZEXPC. When this null hypothesis cannot be rejected the conclusion is that the expected value approach yields a result which is statistically the same as the simulation approach.

The null hypothesis which was tested in each case is that the means of the two populations from which the samples were drawn are equal. The alternative hypothesis is that they are unequal. When the null hypothesis is accepted, the conclusion to be drawn is that, given these conditions, the deterministic equivalent yields a result which is not statistically different from the result determined by the simulation approach.

Analysis of the Experimental Results

For purposes of analysis the experimental results are summarized in Tables 10 through 12 of the appendix. Each table refers to a different phase of the experiment. Within each table the performances of the different deterministic equivalents are summarized for each experimental problem which was used.

The two-stage approach is presented under three different assumptions concerning the cost coefficients in the vectors COSLK and COINF, while the active approach is presented in terms of the two different assumptions concerning the values of the allocation ratios in the U and V matrices.

For each value of the coefficient of variation of the stochastic parameters, an "X" is placed in the appropriate columns of the table if the deterministic equivalent on the average yielded a feasible solution and if the null hypothesis cannot be rejected at either the .05 or the .01 levels of significance. The ith deterministic equivalent is considered to yield a feasible solution "on the average" if the value of Z_i in equation [4] is positive. The value Z_i is positive when $\bar{\bar{d}}_i$ is positive. The reader should realize that even though $\bar{\bar{d}}_i$ is positive this does not exclude the possibility of an infeasible solution for the ith deterministic equivalent resulting on any particular iteration. The $\bar{\bar{d}}_i$ value is determined in the first two phases of the experiment by averaging 400 d_{ij}'s and in the third phase by averaging 900 d_{ij}'s. These d_{ij} values can be positive or negative. The conditional probability of a feasible solution resulting on any iteration increases as the value of Z_i increases from zero. As the value of Z_i decreases from zero, the probability of an infeasible solution on any iteration is increased.

Consider, in Table 10, the data which resulted from using the expected value approach in phase one of the experiment. From the table it can be seen that, given experimental problem A, for a coefficient of variation equal to .05 the expected value approach yielded an infeasible solution on the average which is not statistically different from the solution yielded by the simulation approach at either the .05 or the .01 levels of significance. For experimental problem B the interpretation is the same.

Phase one

The results from phase I of the experiment are summarized in Table 10. The three deterministic equivalents are referred to as ZEXPC, ZTWS, and ZACT. The last two deterministic equivalents mentioned above are presented under multiple assumptions concerning the exogenous parameters used in those approaches.

The expected value approach

In phase one the expected value approach yielded infeasible solutions, on the average, for all values of V considered regardless of the experimental problem used. The results of the Z-test are also similar for both problems, with the only exception occurring when V = .20 and α = .01. At both the .05 and the .01 levels of significance, the null hypothesis was rejected in all cases except where V had small values. This is true for both experimental problems considered.

In summary the expected value approach in phase one yielded infeasible solutions which led to rejection of the hypothesis of no difference at both levels of significance for the larger values of V. In addition this approach was consistent in its results for both experimental problems.

The two-stage approach

The two-stage approach was considered under three different sets of exogenous cost coefficients. In phase one where only the B vector is stochastic the optimum value of the objective function in the two-stage approach is initially the same as the optimum value of the objective function for the expected value approach. The costs resulting from either excess slack or infeasibility in the constraints are then deducted from this initial objective function value. This factor is represented in the table by the fact that the two-stage approach yielded feasible solutions for both problems for the smaller values of V while the expected value approach did not. As the cost coefficients were increased, feasible solutions for both experimental problems resulted for the high values of V.

For the smallest set of cost coefficients the null hypothesis was not rejected at either level of significance for the smaller values of V. For the values of V greater than or equal to .25 the null hypothesis was accepted only at a level of significance of $\alpha = .01$. When the adjustment cost coefficients were increased the results were feasible on the average for the higher level of V. However, as these adjustment cost coefficients were increased, this approach yielded inconsistent results with respect to the tests of the null hypotheses. For example, for the second set of adjustment cost coefficients considered, the null hypothesis was accepted at both levels of significance for all values of V. When the largest set of adjustment cost coefficients were used, the null hypothesis was rejected in all cases except where V = .05 and α = .01. These results were consistent with respect to both of the experimental problems analyzed.

In phase one when the two smallest sets of cost coefficients were used the two-stage approach can be summarized as yielding results which were feasible on the average and not significantly different from the simulation results at the .01 level of significance for all values of V. For the largest set of cost coefficients used the results were significantly different from the simulation results at both of the levels of significance. This approach also yielded very consistent results over both experimental problems considered.

The active approach

The active approach was evaluated under two different sets of allocation ratios. For each experimental problem and for each set of the allocation ratios the solutions using the active approach were feasible on the average for all values of V. For both of the problems analyzed all the tests led to a rejection of the null hypothesis of no difference from the simulation approach for all values of V.

Summary of phase one

In phase one the expected value approach yielded only infeasible solutions on the average. The two-stage approach did yield feasible solutions on the average for the smaller values of V; and as the adjustment cost coefficients increased, feasible solutions resulted for the higher values of V. The active approach always yielded feasible solutions on the average. Each deterministic equivalent yielded consistent results in terms of the feasibility or infeasibility of the results generated for both experimental problems used.

The expected value approach yielded results which were not significantly different from the simulation approach at either level of significance for the smaller values of V. The two-stage approach, utilizing the second set of cost coefficients, yielded results which were not significantly different from the simulation approach at either level of significance for all values of V. For the smallest set of cost coefficients the null hypothesis was rejected at both levels of significance only for the larger values of V, while for the largest set of cost coefficients used, the null hypothesis was rejected at both levels of significance for all values of V. The active approach yielded results which were significantly different from the simulation approach at both levels of significance for all the values of V considered.

Phase two

In phase two of the experiment the stochastic parameters were confined to the C vector. The results of this part of the experiment are presented in Table 11 of the appendix.

The expected value approach

In the second phase of the experiment the expected value approach yielded feasible results on the average in all except the cases dealing with problem A for V = .05 and V = .10. Only in two cases did this approach yield results leading to a rejection of the hypothesis. These exceptions occurred when dealing with problem B for V equal to .25 and α equal to either .05 or .01. As was the case in phase one the results from this approach were consistent over both experimental problems considered.

The two-stage approach

The adjustment cost coefficients of the two-stage approach are related to the constraints of the problem under investigation. Since the stochastic parameters were present only in the C vector in this phase, these adjustment costs do not affect the optimum objective function value which is initially determined. This initial value is determined by postmultiplying the C vector, the elements of which have been randomly determined, by the optimum solution vector of the expected value approach. It was assumed in the experimental model that the investigator, in using this approach, based his planning upon the expected values of the stochastic parameters involved and then later adjusted these plans at an additional cost. As was the case in phase one three sets of adjustment cost coefficients were used in the experimental model.

Since the adjustment cost coefficients are related directly to the constraints and since these constraints are deterministic, then it is expected that the experimental results should be independent of these adjustment costs. Table 11 indicates that this conclusion is correct. There was no change in the experimental results as the adjustment cost coefficients increased.

For any set of adjustment cost coefficients the results of the two-stage approach were feasible on the average for both of the experimental problems analyzed. The tests of the null hypothesis led to a distinct set of conclusions for the two problems considered. In dealing with the slightly constrained problem, the results of the two-stage approach led to a rejection of the null hypothesis at the .05 level of significance for values of V greater than or equal to .20, while at a level of significance of .01 the results led to a rejection of the null hypothesis only when V = .30. The null hypothesis was rejected in all cases at both levels of significance when the tightly constrained problem was analyzed.

The active approach

The active approach in this phase was performed by first converting the experimental problem to its dual problem and then expanding the set of constraints of the dual according to the number of stochastic parameters involved in the formulation. As was the case in phase one the active approach was analyzed under two assumptions concerning the allocation ratios used in the model.

The results of the active approach were consistent with respect to the two sets of allocation ratios used and with respect to the two experimental problems analyzed. For all cases in this phase the results from the active approach were infeasible and led to a rejection of the null hypothesis at both the .05 and the .01 levels of significance.

Summary of phase two

In phase two the expected value approach yielded results which were generally consistent for both types of experimental problems considered. This approach yielded results which were also generally feasible on the average and not significantly different from the simulation results at either level of significance. The two-stage approach yielded feasible results on the average in all cases. These results were significantly different from the simulation results at both levels of significance when dealing with the tightly constrained problem. With the slightly constrained problem the results of this approach were significantly different from the simulation approach only for the higher values of V considered. The active approach yielded results which were consistent for the two experimental problems analyzed. These results were always infeasible on the average and significantly different from the simulation results at both levels of significance for all the values of V considered.

The table also indicates that the results of the two-stage approach were consistent with respect to the different adjustment cost coefficients that were used and that the results of the active approach were consistent with respect to the different allocation ratios used.

Phase three

In phase three of the experimental procedure only the expected value approach and the two-stage approach were evaluated. The two-stage approach was analyzed for each of the three sets of adjustment cost coefficients. In Table 12 the results of the third phase are summarized.

The expected value approach

The results generated by the expected value approach were feasible on the average for both experimental problems only when V = .05. For higher values of V the results were not feasible. For both problems considered the experimental results led to a rejection of the hypothesis at the .05 level of significance only when V equaled .25 or .30. At the level of significance of α = .01 the results for the tightly constrained problem B were not significantly different from the results of the simulation approach for any value of V, whereas for the slightly constrained problem the results were not significantly different for values of V less than or equal to .20. The expected value approach was consistent in its results over the two experimental problems with the only exceptions resulting when V = .25 or .30.

The two-stage approach

The solutions resulting from the two-stage approach in phase three were always feasible on the average for both experimental problems when the two largest sets of adjustment cost coefficients were used. For the smallest set of adjustment cost coefficients the feasibility of the results differed for the two experimental problems. In the slightly constrained problem A the solutions were feasible on the average only for V = .05, while in the tightly constrained problem the solutions were feasible on the average for all V values except V = .30.

The two-stage model utilizing the smallest set of adjustment cost coefficients was consistent for each experimental problem with respect to the rejection of the null hypothesis at both levels of significance tested. The null hypothesis, when problem B was used, was not rejected at either level of significance for any value of V considered. The results when problem A was used led to a rejection of the null hypothesis at the .05 level of significance only when V equaled .30.

As the adjustment cost coefficients increased in this phase, the two-stage approach yielded inconsistent results. For example, when the second set of adjustment cost coefficients were used with problem A, the null hypothesis was accepted at both levels of significance for all the values of V except .05; but when the largest set of adjustment cost coefficients were used with the same problem, the null hypothesis was rejected at both of the levels of significance for all the values of V. Correspondingly the results also differed when the adjustment cost coefficients were increased in dealing with the tightly constrained problem. In this phase it appeared that the adjustment cost coefficients had a direct bearing upon the acceptance or the rejection of the null hypothesis for both of the types of problems considered.

Summary of phase three

In this phase the expected value approach generally yielded results which were not feasible on the average. In addition, these results generally led to acceptance of the null hypothesis of no difference from the simulation results at both levels of significance for all values of V except for V greater than or equal to .25.

For the smallest set of adjustment cost coefficients the two-stage approach yielded results which generally led to an acceptance of the null hypothesis at both level of significance for both problems considered. Only when dealing with the tightly constrained problem were the results generally feasible on the average for this set of cost coefficients. As the adjustment cost coefficients were increased the results became feasible on the average for both problems considered and the results generally led to a rejection of the null hypothesis in all cases except those dealing with the second set of cost coefficients with the slightly constrained problem.

Conclusions

The development of an experimental model which can be used to evaluate proposed deterministic equivalents to the stochastic programming model was an important result of this study. The model as it was used in the study evaluated some of the linear deterministic equivalents to the stochastic programming model.

Before a statement of the findings from this study is presented it is necessary to briefly review the assumptions upon which the experimental model has been built. These major assumptions are as follows. (1) The stochastic parameters which appear in each formulation are assumed to be normally and independently distributed with known means and variances. (2) A specific empirical problem is used as a means of generating the results of each deterministic equivalent and of the simulation approach. This initial problem is a slightly constrained problem. (3) A modified form of the initial problem is used as an example of a tightly constrained problem and another set of results are generated. (4) The expected value solution is used to determine a ranking of the constraints and of the variables of the experimental problem. These rankings are used to select appropriate formulations of the two experimental problems which are analyzed in the experimental model. (5) Each deterministic equivalent is evaluated as the positions of the stochastic parameters change and as the variances of the stochastic parameters change.

Major findings

The major findings are summarized for each phase of the experiment. In the first phase the stochastic parameters are limited to the B vector; in the second phase they are limited to the C vector; and in the third phase the stochastic parameters appear in both vectors.

Phase one

The results indicate that the two-stage approach is the best deterministic equivalent to use in phase one. This approach is only slightly affected as the variances of the stochastic parameters increase. The results of this approach are also consistent for the two types of problems considered.

It should be pointed out that these conclusions are dependent upon the adjustment cost coefficients which are used. The results of the two-stage approach are affected by changes in the values of the adjustment cost coefficients.

The expected value approach does not on the average yield feasible solutions and becomes unreliable as the variances of the stochastic parameters increase. As a first approximation, however, the expected value approach does have some advantages, particularly when it is used to generate an initial solution in the two-stage approach.

The least desirable approach in this phase is the active approach. In all cases this approach yields feasible results on the average; however, these results are statistically different from the results generated by the simulation approach. This approach is conservative in that it limits the optimum value of the objective function by restricting the use of the resources through the allocation ratios. In addition the results of the active approach are consistent for each problem analyzed and for each set of allocation ratios used.

Phase two

The expected value approach is the best approach to use when the stochastic parameters appear only in the C vector. This approach yields feasible solutions on the average which are very reliable regardless of the type of problem analyzed. In addition the approach is not affected by increases in the variances of the stochastic parameters.

The two-stage approach is the next best approach in this phase. From the results it can be seen that the adjustment costs have no affect in this phase. This is to be expected since these costs are associated only with the constraints which are deterministic. The two-stage approach always yields a feasible solution on the average and is more reliable when dealing with the slightly constrained problem than when dealing with the tightly constrained problem. When dealing with the tightly constrained problem this approach was affected somewhat by increasing the variances of the stochastic parameters.

The active approach is the least desirable approach in this phase since it always yields infeasible solutions on the average which are very unreliable regardless of the type of experimental problem considered.

Phase three

In the third phase, of the two deterministic equivalents evaluated, the two-stage approach is considered the better since it yields feasible solutions on the average in more cases than does the expected value approach. The results of this approach are generally statistically the same as the results of the simulation approach. However these results are affected by the different values of the adjustment cost coefficients which are used. As the cost coefficients increase the results tend to become feasible on the average but also tend to become significantly different from the results of the simulation approach.

The results of the expected value approach are affected by increasing the variances of the stochastic parameters. For only the smallest set of values of the variances does this approach yield results which are feasible on the average. The results of this approach are generally the same as the results of the simulation approach except for the two largest sets of values of the variances of the stochastic parameters. In addition the approach is consistent with respect to the two problems considered.

Areas of Further Research

During the development of the experimental model and the analysis of the results from the experiment a number of questions arose which can serve as the basis upon which additional experimentation can be performed. Some of the more important areas for further research are as follows. (1) The deterministic equivalents can be studied assuming nonnormal distribution for the stochastic parameters. In addition formulations of an experimental problem can be studied where the different parameters are distributed according to different types of distributions. (2) The distributions of the optimum objective function values which result from the application of the different deterministic equivalents can be analyzed to determine the properties of these distributions and the effects that the properties of the stochastic parameters have upon these distributions. (3) Additional deterministic equivalents, some of which are presented in the second chapter, can be analyzed by the experimental model to determine the effects that the positions and the properties of the stochastic parameters have upon the performance of these equivalents. (4) The possibility of combining different deterministic equivalents into one model in order to utilize the advantages of each can be investigated. For example the expected value approach is used in this way with the two-stage deterministic equivalent. Since the expected value approach appears to be a good first approximation, the possibility of combining it with other deterministic equivalents can be studied. (5) The results of the active approach are dependent upon the specific values of the allocation ratios used in the model. The effects that these ratios have upon the results generated by this deterministic equivalent can be analyzed. For example, the use of different sets of allocation ratios for a tightly constrained problem and a slightly constrained problem can be studied. (6) The adjustment cost coefficients have an effect upon the results of the two-stage approach. The effects that these coefficients have can be studied. For example in the first and the third phases of the experiment the results of the two-stage approach varied significantly as the cost coefficients changed.

APPENDIX

TABLE 2

THE SAMPLE MEANS OF THE OPTIMUM OBJECTIVE FUNCTION VALUES FOR ALL EXPERIMENTS IN PHASE I PROBLEM A

V (σ/μ)	EXP. NO.	ZSIMBR	ZEXPC	ZTWSBR (20)	ZTWSBR (40)	ZTWSBR (80)	ZACTBR (.90)	ZACTBR (.75)
.05	1	8842.14	8837.96	8834.97	8831.98	8825.98	8324.25	7429.36
	2	8896.48	8837.96	8777.25	8716.54	8595.11	8261.65	7339.39
	3	8708.01	8837.96	8826.90	8815.83	8793.70	8089.88	7032.54
	4	8845.02	8837.96	8774.53	8711.09	8584.20	8120.71	7034.26
.10	5	8820.29	8837.96	8831.99	8826.01	8814.06	8316.83	7422.90
	6	8756.02	8837.96	8677.09	8516.21	8194.45	8132.05	7196.11
	7	8689.84	8837.96	8813.01	8788.05	8738.14	8092.89	7040.04
	8	8791.73	8837.96	8662.68	8487.38	8136.80	8074.16	6997.82
.15	9	8702.29	8837.96	8827.45	8816.92	8795.88	8266.22	7397.07
	10	8700.42	8837.96	8552.79	8267.61	7697.25	8089.08	7172.07
	11	8703.57	8837.96	8800.89	8763.62	8689.67	8147.72	7101.84
	12	8704.07	8837.96	8537.06	8236.16	7634.36	7995.77	6933.33
.20	13	8702.75	8837.96	8824.16	8810.35	8782.73	8355.50	7547.66
	14	8294.43	8837.96	8450.21	8062.45	7286.94	7709.18	6831.32
	15	8608.89	8837.96	8793.20	8748.44	8658.91	8122.80	7090.62
	16	8332.28	8837.96	8361.82	7885.67	6933.38	7659.73	6650.91
.25	17	8608.08	8837.96	8823.72	8809.47	8780.97	8258.96	7456.60
	18	7675.27	8837.96	8198.45	7558.94	6279.92	7141.65	6341.21
	19	8141.15	8837.96	8779.66	8721.36	8604.77	7680.37	6775.84
	20	7993.59	8837.96	8226.90	7615.83	6393.70	7358.46	6405.76
.30	21	8328.71	8837.96	8819.35	8800.73	8763.50	8016.16	7275.88
	22	7765.91	8837.96	8123.21	7408.45	5978.93	7239.64	6419.97
	23	8162.20	8837.96	8758.96	8679.95	8521.93	7728.00	6849.71
	24	7342.79	8837.96	7961.36	7084.75	5331.54	6761.45	5889.44

TABLE 3

THE SAMPLE MEANS OF THE OPTIMUM OBJECTIVE FUNCTION VALUES FOR ALL EXPERIMENTS IN PHASE II PROBLEM A

V (σ/μ)	EXP. NO.	ZSIMBR	ZEXPC	ZTWSBR (20)	ZTWSBR (40)	ZTWSBR (80)	ZACTBR (.90)	ZACTBR (.75)
.05	25	8908.52	8837.96	8908.52	8908.52	8908.52	9249.44	9760.82
	26	8837.66	8837.96	8837.66	8837.66	8837.66	9175.87	9683.18
	27	8782.91	8837.96	8782.91	8782.91	8782.91	9119.02	9623.19
	28	8809.51	8837.96	8809.51	8809.51	8809.51	9146.63	9652.33
.10	29	8812.71	8837.96	8812.70	8812.70	8812.70	9149.95	9655.82
	30	8726.16	8837.96	8726.05	8726.05	8726.05	9068.64	9568.10
	31	8881.23	8837.96	8881.18	8881.18	8881.18	9221.05	9730.86
	32	8822.34	8837.96	8822.15	8822.15	8822.15	9173.95	9678.00
.15	33	8783.38	8837.96	8782.64	8782.64	8782.64	9118.74	9622.89
	34	8853.25	8837.96	8852.04	8852.04	8852.04	9281.65	9774.64
	35	9027.44	8837.96	9022.17	9022.17	9022.17	9385.21	9900.14
	36	8816.38	8837.96	8807.80	8807.80	8807.80	9260.22	9746.59
.20	37	8858.18	8837.96	8854.63	8854.63	8854.63	9194.66	9701.77
	38	8933.11	8837.96	8929.59	8929.59	8929.59	9440.33	9924.75
	39	8686.31	8837.96	8677.31	8677.31	8677.31	9035.24	9529.04
	40	9141.80	8837.96	9131.85	9131.85	9131.85	9749.32	10228.85
.25	41	9292.91	8837.96	9278.13	9278.13	9278.13	9647.19	10169.52
	42	9471.45	8837.96	9469.24	9469.24	9469.24	9986.00	10503.84
	43	8756.77	8837.96	8714.73	8714.73	8714.73	9187.98	9664.88
	44	9017.10	8837.96	9002.54	9002.54	9002.54	9623.71	10094.64
.30	45	8898.95	8837.96	8855.91	8855.91	8855.91	9251.43	9742.54
	46	8952.70	8837.96	8924.36	8924.36	8924.36	9682.74	10125.79
	47	8507.64	8837.96	8429.02	8429.02	8429.02	9002.52	9453.71
	48	9344.43	8837.96	9285.29	9285.29	9285.29	10204.07	10649.10

TABLE 4

THE SAMPLE MEANS OF THE OPTIMUM OBJECTIVE FUNCTION VALUES FOR ALL EXPERIMENTS IN PHASE III PROBLEM A

V (σ/μ)	EXP. NO.	ZSIMBR	ZEXPC	ZTWSBR (20)	ZTWSBR (40)	ZTWSBR (80)
.05	49	8873.36	8837.96	8824.28	8820.44	8812.77
	50	8906.91	8837.96	8795.74	8793.03	8787.61
	51	8845.32	8837.96	8748.13	8682.32	8550.71
	52	8853.83	8837.96	8876.89	8873.65	8867.17
	53	8817.57	8837.96	8821.83	8754.07	8618.54
	54	8865.68	8837.96	8767.53	8712.57	8602.66
	55	8866.89	8837.96	8838.43	8774.58	8646.88
	56	8942.45	8837.96	8780.13	8705.55	8556.40
	57	8894.40	8837.96	8803.21	8744.73	8627.76
.10	58	8805.82	8837.96	8802.38	8796.05	8783.38
	59	8613.24	8837.96	8802.64	8795.65	8781.66
	60	8837.61	8837.96	8740.24	8606.73	8339.71
	61	8913.39	8837.96	8938.15	8931.93	8919.50
	62	8475.67	8837.96	8485.83	8304.95	7943.20
	63	8629.26	8837.96	8620.13	8456.69	8129.81
	64	8750.25	8837.96	8709.48	8564.83	8275.52
	65	8758.57	8837.96	8772.66	8590.67	8226.70
	66	8619.03	8837.96	8563.73	8389.09	8039.80
.15	67	8879.04	8837.96	8923.76	8914.93	8897.27
	68	8853.49	8837.96	8802.48	8792.95	8773.88
	69	8699.75	8837.96	8630.36	8372.74	7857.50
	70	8866.36	8837.96	8844.88	8836.16	8818.74
	71	8364.64	8837.96	8585.95	8311.07	7761.29
	72	8648.34	8837.96	8750.60	8494.27	7981.60
	73	8582.41	8837.96	8471.21	8179.49	7596.06
	74	8538.84	8837.96	8673.26	8400.85	7846.04
	75	8349.65	8837.96	8399.77	8095.27	7486.27

(CONTINUED)

TABLE 4-CONTINUED

V (σ/μ)	EXP. NO.	ZSIMBR	ZEXPC	ZTWSBR (20)	ZTWSBR (40)	ZTWSBR (80)
.20	76	8313.55	8837.96	8706.69	8693.45	8666.97
	77	8477.64	8837.96	8721.25	8706.84	8678.02
	78	8097.00	8837.96	8474.01	8044.57	7185.69
	79	8311.57	8837.96	8803.37	8787.55	8755.92
	80	8695.24	8837.96	8701.91	8168.57	7101.88
	81	8777.83	8837.96	8724.51	8232.38	7248.13
	82	8041.31	8837.96	7965.73	7442.17	6395.04
	83	8414.81	8837.96	8564.41	8124.58	7244.92
	84	8556.81	8837.96	8580.57	8079.46	7077.23
.25	85	8547.16	8837.96	8764.19	8747.14	8713.03
	86	8501.03	8837.96	8820.74	8805.32	8774.47
	87	7554.36	8837.96	8047.45	7487.96	6368.99
	88	8075.09	8837.96	8647.65	8630.70	8596.79
	89	7585.52	8837.96	7727.73	7009.38	5572.67
	90	8582.09	8837.96	8510.31	7879.52	6617.92
	91	8057.27	8837.96	8084.69	7526.49	6410.09
	92	7929.35	8837.96	8075.74	7580.52	6590.09
	93	7681.05	8837.96	7923.52	7299.07	6050.16
.30	94	8691.92	8837.96	8925.32	8905.86	8866.93
	95	8660.93	8837.96	8889.29	8871.97	8837.35
	96	7236.34	8837.96	8195.89	7372.88	5726.85
	97	8596.45	8837.96	8903.33	8886.00	8851.32
	98	7362.34	8837.96	7815.34	7006.01	5387.36
	99	7664.05	8837.96	8098.05	7317.56	5756.58
	100	7489.18	8837.96	7887.97	7218.98	5880.99
	101	7392.61	8837.96	7735.71	6905.35	5244.63
	102	7591.15	8837.96	8042.77	7354.28	5977.30

TABLE 5

THE SAMPLE MEANS OF THE OPTIMUM OBJECTIVE FUNCTION VALUES
FOR ALL EXPERIMENTS IN PHASE I PROBLEM B

V (σ/μ)	EXP. NO.	ZSIMBR	ZEXPC	ZTWSBR (20)	ZTWSBR (40)	ZTWSBR (80)	ZACTBR (.90)	ZACTBR (.75)
.05	1	9137.01	9133.07	9130.07	9127.07	9121.09	8954.73	8972.04
	2	9176.50	9133.07	9070.77	9008.48	8883.90	8907.00	8889.64
	3	9013.87	9133.07	9111.93	9090.80	9048.53	8283.73	7647.32
	4	9112.39	9133.07	9048.32	8963.58	8794.09	8401.48	7736.17
.10	5	9098.79	9133.07	9127.09	9121.11	9109.16	8916.21	8894.00
	6	8999.96	9133.07	8949.02	8764.97	8396.88	8759.00	8708.21
	7	8983.73	9133.07	9082.34	9031.60	8930.14	8286.31	7661.09
	8	9044.78	9133.07	8916.13	8699.19	8265.32	8356.72	7704.21
.15	9	8963.83	9133.07	9122.54	9112.02	9090.98	8780.91	8738.62
	10	8944.93	9133.07	8820.18	8507.29	7881.52	8719.43	8628.79
	11	8993.25	9133.07	9058.59	8984.11	8835.14	8340.93	7732.50
	12	8947.76	9133.07	8766.03	8399.00	7664.93	8278.11	7639.16
.20	13	8936.61	9133.07	9119.26	9105.45	9077.83	8766.54	8755.55
	14	8547.74	9133.07	8705.19	8277.32	7421.57	8304.46	8201.88
	15	8684.92	9133.07	9047.69	8962.32	8791.57	8291.51	7723.92
	16	8582.68	9133.07	8564.27	7995.46	6857.86	7940.37	7352.51
.25	17	8854.46	9133.07	9118.82	9104.57	9076.07	8676.13	8663.84
	18	7933.91	9133.07	8446.64	7760.21	6387.36	7693.04	7611.92
	19	8408.93	9133.07	8997.26	8861.46	8589.85	7838.19	7317.58
	20	8236.90	9133.07	8421.19	7709.31	6285.56	7651.61	7138.63
.30	21	8578.07	9133.07	9114.45	9095.83	9058.59	8390.93	8379.73
	22	8023.11	9133.07	8367.60	7602.13	6071.19	7774.50	7557.19
	23	8486.41	9133.07	8998.17	8863.27	8593.48	7899.69	7458.08
	24	7609.27	9133.07	8117.04	7101.02	5068.96	7030.23	6561.40

TABLE 6

THE SAMPLE MEANS OF THE OPTIMUM OBJECTIVE FUNCTION VALUES
FOR ALL EXPERIMENTS IN PHASE II PROBLEM B

V (σ/μ)	EXP. NO.	ZSIMBR	ZEXPC	ZTWSBR (20)	ZTWSBR (40)	ZTWSBR (80)	ZACTBR (.90)	ZACTBR (.75)
.05	25	9188.80	9133.07	9188.80	9188.80	9188.80	9326.36	9760.82
	26	9141.82	9133.07	9123.65	9123.65	9123.65	10422.34	10749.05
	27	9110.64	9133.07	9089.58	9089.58	9089.58	9958.81	10319.91
	28	9142.28	9133.07	9090.44	9090.44	9090.44	10490.00	10799.63
.10	29	9120.75	9133.07	9113.11	9113.11	9113.11	9345.34	9660.70
	30	9083.36	9133.07	9006.10	9006.10	9006.10	10389.07	10713.46
	31	9241.83	9133.07	9167.20	9167.20	9167.20	10182.07	10518.32
	32	9232.64	9133.07	9086.87	9086.87	9086.87	10782.33	11063.26
.15	33	9112.01	9133.07	9089.36	9089.36	9089.36	9384.96	9678.02
	34	9255.62	9133.07	9118.85	9118.85	9118.85	10709.12	11028.52
	35	9464.68	9133.07	9278.57	9278.57	9278.57	10666.88	10966.76
	36	9270.64	9133.07	9060.18	9060.18	9060.18	11003.03	11279.23
.20	37	9198.57	9133.07	9146.23	9146.23	9146.23	9522.59	9808.45
	38	9413.37	9133.07	9220.42	9220.42	9220.42	11070.96	11338.70
	39	9220.60	9133.07	9006.16	9006.16	9006.16	10643.74	10852.49
	40	9670.14	9133.07	9409.00	9409.00	9409.00	11655.76	11918.40
.25	41	9590.25	9133.07	9480.75	9480.75	9480.75	9942.71	10305.32
	42	9860.71	9133.07	9560.49	9560.49	9560.49	11258.05	11620.91
	43	9392.41	9133.07	9035.72	9035.72	9035.72	11159.91	11300.53
	44	9636.59	9133.07	9205.24	9205.24	9205.24	11824.98	12040.11
.30	45	9313.63	9133.07	9147.24	9147.24	9147.24	9767.11	10030.36
	46	9475.66	9133.07	9198.78	9198.78	9198.78	11282.48	11522.05
	47	9173.41	9133.07	8810.04	8810.04	8810.04	10941.08	11043.11
	48	9939.57	9133.07	9462.86	9462.86	9462.86	12214.27	12457.14

TABLE 7

THE SAMPLE MEANS OF THE OPTIMUM OBJECTIVE FUNCTION VALUES FOR ALL EXPERIMENTS IN PHASE III PROBLEM B

V (σ/μ)	EXP. NO.	ZSIMBR	ZEXPC	ZTWSBR (20)	ZTWSBR (40)	ZTWSBR (80)
.05	49	9166.49	9133.07	9121.45	9117.61	9109.95
	50	9220.23	9133.07	9097.92	9095.21	9089.79
	51	9128.91	9133.07	9043.54	8973.00	8831.91
	52	9163.41	9133.07	9160.24	9157.00	9150.52
	53	9100.19	9133.07	9091.92	9010.00	8846.14
	54	9167.70	9133.07	9067.79	9008.59	8890.20
	55	9173.93	9133.07	9108.33	9043.99	8915.32
	56	9241.19	9133.07	9065.21	8976.55	8799.23
	57	9201.83	9133.07	9079.13	9005.14	8857.18
.10	58	9099.20	9133.07	9103.63	9097.30	9084.63
	59	8989.88	9133.07	9123.36	9116.36	9102.38
	60	9077.90	9133.07	8999.91	8838.49	8515.64
	61	9256.27	9133.07	9197.97	9191.75	9179.32
	62	8788.94	9133.07	8777.92	8558.05	8118.31
	63	8971.65	9133.07	8923.23	8731.84	8349.04
	64	9096.97	9133.07	8970.41	8300.50	8460.68
	65	9074.93	9133.07	9003.05	8780.73	8336.11
	66	8939.19	9133.07	8803.44	8561.46	8077.50
.15	67	9167.56	9133.07	9198.98	9190.15	9172.49
	68	9251.32	9133.07	9115.15	9105.61	9086.55
	69	8963.14	9133.07	8887.13	8601.68	8030.80
	70	9302.17	9133.07	9166.57	9157.86	9140.43
	71	8683.69	9133.07	8808.20	8465.27	7779.40
	72	8984.22	9133.07	8937.81	8651.36	8078.51
	73	8985.64	9133.07	8748.63	8421.71	7767.88
	74	8945.23	9133.07	8912.18	8572.00	7891.64
	75	8772.02	9133.07	8619.57	8221.54	7425.46

(CONTINUED)

TABLE 7–CONTINUED

V (σ/μ)	EXP. NO.	ZSIMBR	ZEXPC	ZTWSBR (20)	ZTWSBR (40)	ZTWSBR (80)
.20	76	8678.95	9133.07	9026.59	9013.35	8986.88
	77	8937.75	9133.07	9063.11	9048.70	9019.88
	78	8425.00	9133.07	8716.59	8248.40	7312.02
	79	8919.55	9133.07	9137.48	9121.66	9090.03
	80	8979.48	9133.07	8817.58	8188.28	6929.68
	81	9153.12	9133.07	8901.58	8366.95	7297.67
	82	8623.34	9133.07	8258.45	7698.61	6578.94
	83	8865.39	9133.07	8795.91	8258.57	7183.89
	84	8968.09	9133.07	8716.42	8112.73	6905.34
.25	85	8931.69	9133.07	9071.21	9054.15	9020.05
	86	9027.52	9133.07	9159.36	9143.94	9113.09
	87	7977.75	9133.07	8332.95	7715.33	6480.08
	88	8792.13	9133.07	8961.73	8944.78	8910.87
	89	7975.56	9133.07	7994.39	7165.26	5507.00
	90	9077.77	9133.07	8713.45	8032.23	6669.79
	91	8672.82	9133.07	8367.05	7751.40	6520.11
	92	8478.36	9133.07	8294.60	7676.04	6438.92
	93	8344.67	9133.07	8170.89	7420.36	5919.27
.30	94	9048.35	9133.07	9197.98	9178.52	9139.59
	95	9116.69	9133.07	9134.80	9117.49	9082.87
	96	7631.45	9133.07	8394.65	7513.30	5750.61
	97	9244.91	9133.07	9227.49	9210.16	9175.43
	98	7882.11	9133.07	8036.72	7108.86	5253.14
	99	8237.15	9133.07	8282.29	7440.78	5757.76
	100	8291.58	9133.07	8227.70	7501.31	6048.52
	101	8010.80	9133.07	7939.69	6993.68	5101.64
	102	8251.77	9133.07	8239.64	7445.29	5856.59

TABLE 8

TEST RESULTS ON THE VARIOUS DETERMINISTIC EQUIVALENTS PROBLEM A

V (σ/μ)	EXPECTED VALUE	TWO-STAGE (20)	TWO-STAGE (40)	TWO-STAGE (80)	ACTIVE (.90)	ACTIVE (.75)
PHASE I						
.05	-0.332	0.430	1.173	2.474	112.202	136.565
.10	-0.899	0.226	1.340	3.212	60.588	77.799
.15	-1.172	0.196	1.469	3.439	39.390	54.986
.20	-2.495	-0.852	0.730	3.309	28.383	40.511
.25	-4.005	-2.327	-0.410	2.765	24.940	30.806
.30	-4.244	-2.419	-0.408	2.846	21.279	25.930
PHASE II						
.05	-0.073	0.0	0.0	0.0	-195.763	-195.505
.10	-0.324	1.360	1.360	1.360	-64.981	-101.656
.15	0.251	1.462	1.462	1.462	-18.924	-48.486
.20	0.379	2.475	2.475	2.475	-11.980	-30.646
.25	1.395	2.269	2.269	2.269	-10.310	-24.943
.30	0.337	2.805	2.805	2.805	-7.496	-16.947
PHASE III						
.05	0.585	1.501	2.367	3.707	NA	NA
.10	-1.067	-0.044	1.256	3.285	NA	NA
.15	-1.098	-0.281	1.239	3.657	NA	NA
.20	-1.888	-1.105	0.967	4.155	NA	NA
.25	-2.834	-1.259	0.902	4.206	NA	NA
.30	-3.099	-1.979	0.425	4.009	NA	NA

CRITICAL VALUES OF Z FOR α = .05 ARE $\pm$1.96
FOR α = .01 ARE $\pm$2.58

TABLE 9

TEST RESULTS ON THE VARIOUS DETERMINISTIC EQUIVALENTS PROBLEM B

V (σ/μ)	EXPECTED VALUE	TWO-STAGE (20)	TWO-STAGE (40)	TWO-STAGE (80)	ACTIVE (.90)	ACTIVE (.75)
PHASE I						
.05	-0.548	0.465	1.443	3.103	109.216	74.072
.10	-1.339	0.179	1.708	4.184	50.418	47.920
.15	-1.585	0.189	1.849	4.333	31.488	33.072
.20	-2.894	-0.917	1.129	4.311	24.299	25.297
.25	-4.430	-2.398	-0.002	3.797	21.404	20.008
.30	-4.541	-2.387	0.042	3.800	20.469	18.239
PHASE II						
.05	0.359	4.088	4.088	4.088	-22.609	-39.346
.10	0.539	5.401	5.401	5.401	-15.582	-24.349
.15	1.375	5.691	5.691	5.691	-11.766	-17.847
.20	1.645	6.123	6.123	6.123	-11.538	-16.372
.25	2.677	7.609	7.609	7.609	-10.835	-15.395
.30	1.535	7.310	7.310	7.310	-9.778	-13.082
PHASE III						
.05	0.757	1.894	2.910	4.417	NA	NA
.10	-0.969	0.570	2.248	4.682	NA	NA
.15	-0.905	0.646	2.521	5.247	NA	NA
.20	-1.454	0.090	2.587	6.036	NA	NA
.25	-2.223	0.135	2.673	6.165	NA	NA
.30	-2.515	-0.518	2.170	5.743	NA	NA

CRITICAL VALUES OF Z FOR α = .05 ARE ±1.96
FOR α = .01 ARE ±2.58

TABLE 10

SUMMARY OF THE EXPERIMENTAL RESULTS - PHASE I

		PROBLEM A			PROBLEM B		
MODEL	V (σ/μ)	FEASIBLE	ACCEPT H_o .05	ACCEPT H_o .01	FEASIBLE	ACCEPT H_o .05	ACCEPT H_o .01
ZEXPC	.05		X	X		X	X
	.10		X	X		X	X
	.15		X	X		X	X
	.20			X			
	.25						
	.30						
ZTWS(20)	.05	X	X	X	X	X	X
	.10	X	X	X	X	X	X
	.15	X	X	X	X	X	X
	.20		X	X		X	X
	.25			X			X
	.30			X			X
ZTWS(40)	.05	X	X	X	X	X	X
	.10	X	X	X	X	X	X
	.15	X	X	X	X	X	X
	.20	X	X	X	X	X	X
	.25		X	X		X	X
	.30		X	X	X	X	X
ZTWS(80)	.05	X		X	X		
	.10	X			X		
	.15	X			X		
	.20	X			X		
	.25	X			X		
	.30	X			X		
ZACT(.90)	.05	X			X		
	.10	X			X		
	.15	X			X		
	.20	X			X		
	.25	X			X		
	.30	X			X		
ZACT(.75)	.05	X			X		
	.10	X			X		
	.15	X			X		
	.20	X			X		
	.25	X			X		
	.30	X			X		

TABLE 11

SUMMARY OF THE EXPERIMENTAL RESULTS - PHASE II

		PROBLEM A			PROBLEM B		
MODEL	V (σ/μ)	FEASIBLE	ACCEPT H_o .05	ACCEPT H_o .01	FEASIBLE	ACCEPT H_o .05	ACCEPT H_o .01
ZEXPC	.05		X	X	X	X	X
	.10		X	X	X	X	X
	.15	X	X	X	X	X	X
	.20	X	X	X	X	X	X
	.25	X	X	X	X		
	.30	X	X	X	X	X	X
ZTWS(20)	.05	X	X	X	X		
	.10	X	X	X	X		
	.15	X	X	X	X		
	.20	X		X	X		
	.25	X		X	X		
	.30	X			X		
ZTWS(40)	.05	X	X	X	X		
	.10	X	X	X	X		
	.15	X	X	X	X		
	.20	X		X	X		
	.25	X		X	X		
	.30	X			X		
ZTWS(80)	.05	X	X	X	X		
	.10	X	X	X	X		
	.15	X	X	X	X		
	.20	X		X	X		
	.25	X		X	X		
	.30	X			X		
ZACT(.90)	.05						
	.10						
	.15						
	.20						
	.25						
	.30						
ZACT(.75)	.05						
	.10						
	.15						
	.20						
	.25						
	.30						

TABLE 12

SUMMARY OF THE EXPERIMENTAL RESULTS - PHASE III

		PROBLEM A			PROBLEM B		
MODEL	V (σ/μ)	FEASI-BLE	ACCEPT H_o .05	ACCEPT H_o .01	FEASI-BLE	ACCEPT H_o .05	ACCEPT H_o .01
ZEXPC	.05	X	X	X	X	X	X
	.10		X	X		X	X
	.15		X	X		X	X
	.20		X	X		X	X
	.25						X
	.30						X
ZTWS(20)	.05	X	X	X	X	X	X
	.10		X	X	X	X	X
	.15		X	X	X	X	X
	.20		X	X	X	X	X
	.25		X	X	X	X	X
	.30			X		X	X
ZTWS(40)	.05	X		X	X		
	.10	X	X	X	X		X
	.15	X	X	X	X		X
	.20	X	X	X	X		
	.25	X	X	X	X		
	.30	X	X	X	X		X
ZTWS(80)	.05	X			X		
	.10	X			X		
	.15	X			X		
	.20	X			X		
	.25	X			X		
	.30	X			X		

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Major Field: Quantitative Methods

Title of Thesis: An Appraisal of the Efficiency of Alternative Deterministic Equivalents to the Stochastic Programming Model

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